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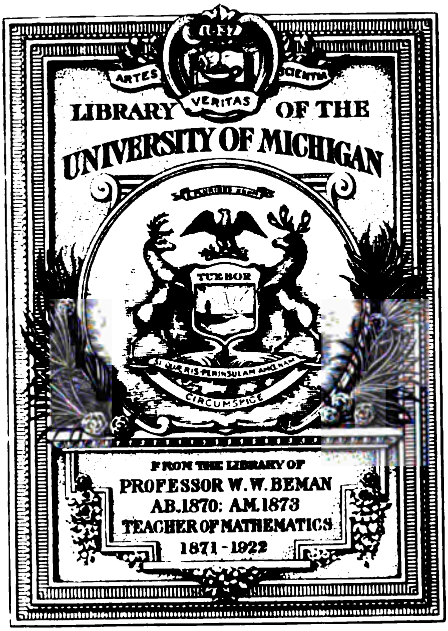

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ELEMENTS

OF

PLANE TRIGONOMETRY.

In which is introduced,

A DISSERTATION

ON THE

NATURE and USE of LOGARITHMS.

1731-1824

By FRANCIS MASERES, M. A.
of CLARE-HALL, CAMBRIDGE.

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W. W. Beman
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6-19-1923

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PREFACE.

THE following treatiſe of TRIGONOMETRY is written with a view to make the attainment of that uſeful ſcience as eaſy to young beginners as the nature of it will permit; the common treatiſes on this ſubject being, as 'tis conceived, from their too great brevity, and other circumſtances, not entirely fit to anſwer that purpoſe. This work is divided into three parts, or books, that differ conſiderably from each other in the nature of the things they treat of, and accordingly require different degrees of previous knowledge in the mathematics to underſtand them: it will therefore be proper in this place to give the reader ſome account of them, that he may be able to ſelect ſuch parts as he is moſt inclined, or beſt prepared, to become acquainted with.

The firſt part contains the doctrine of TRIGONOMETRY properly ſo called, or an explanation of the method of computing a table of ſines and tangents by common geometry, without the help of infinite ſerieſes, and of ſolving the ſeveral caſes of plane triangles by the means of a table ſo computed. This part is nothing more than a copious paraphraſe on Dr. *Keill's* Elements of Trigonometry, with a number of examples of the calculations of ſines and tangents actually performed, and the operations preſented at full length to the reader's view; which to me appeared abſolutely neceſſary to give young beginners in theſe ſtudies a clear and familiar notion of the manner in which theſe tables are computed. The knowledge requiſite to underſtand this part of the work is that of the firſt ſix books of *Euclid*'s Elements, and the operations of common arithmetic, with the principles on which they are founded. The ſcholium indeed in page 33. requires ſome knowledge of algebra and infinite ſerieſes; but as it is quite independent of all that immediately precedes and follows it, the reader that is unacquainted with thoſe ſubjects may paſs it over without the leaſt inconvenience.

The ſecond part of thoſe elements is a collection of ſome of the moſt remarkable properties of ſines and tangents and the other lines belonging to a circle, many of which are of frequent uſe in the ſeveral branches of mathematical philoſophy. This part may be underſtood by the help of the firſt ſix books of *Euclid*'s Elements, and the common principles and operations of algebra; or, if we except the 25th and 26th propoſitions, and the corollaries to prop. 27. (which are not neceſſary towards underſtanding the remaining propoſitions of this ſecond part) by the help of *Euclid's* Elements only. In the 26th propoſition I have endeavoured to give a clear account of the doctrine of the ſines of multiple arcs, and to explain what is meant by ſuch a ſine's becoming negative, or, as the algebriſts expreſs it, paſſing from affirmation through nothing into negation, which, being an obſcure and myſterious kind of expreſſion, required a copious explanation. And this is the principal reaſon of the great length of this propoſition and its corollaries, which take up fourſcore pages of the book; for they could not, without omitting ſomething material to the purpoſe, be reduced into a narrower compaſs. The great quantity of calculation in this propoſition and its corollaries, and the number of algebraic expreſſions uſed in them, give them at firſt ſight an air of difficulty: but it will be found upon examination, that they are in reality a great deal the eaſier for this unfavourable appearance: for this is owing merely to the ſetting down the reaſonings and operations all at length, which the reader muſt have ſupplied by his own induſtry, if they had been omitted for the ſake of making the demonſtrations appear more ſuccinct and elegant. And for the ſame reaſon it is true in general of all books of algebra, that (if the ſubject and method of demonſtration are the ſame) the more calculation appears in them, the eaſier they will be underſtood.

The third part of theſe elements treats of the fluxions of ſines and tangents, and the other lines belonging to a circle, and contains an account of the relations of thoſe ſeveral lines to the circular arc to which they belong; which relations, 'tis well known, cannot be diſcovered but by the method of fluxions. This part is ſubdivided into two others, in the former of which is introduced a diſſertation upon logarithms.

rithms. The whole of this firſt ſubdiviſion, together with the diſſertation on logarithms, is written in the ſame full and explanatory manner as the reſt of the book, and may be underſtood by the help of *Euclid*'s elements, the preceding part of this treatiſe, the property of the aſymptotes of an hyperbola, and the common principles and operations of algebra; or, if we except the diſſertation on logarithms, without any knowledge of the hyperbola or of algebra. This firſt ſubdiviſion contains an account of the proportion of the fluxions of ſines and tangents, and the other lines belonging to a circle, to the fluxion of the arch, and to each other. The latter ſubdiviſion of this part of the work, entitled the Concluſion, contains an account of the ſeveral ſerieſes relating both to circular arcs and logarithms, which are deduced from the principles laid down in the former ſubdiviſion and the diſſertation on logarithms, by the methods of fluxions and the reverſion of ſerieſes. This part of the work therefore cannot be underſtood without ſome knowledge of thoſe methods: and accordingly, as the readers of it muſt be ſuppoſed to be pretty far advanced in theſe ſtudies, it is written with much greater brevity than any of the preceding parts of thoſe elements.

Thoſe who are not inclined to read the diſſertation on logarithms may paſs it over without any inconvenience, and proceed to page 406, coroll. 11. This diſſertation contains a very full account of the nature of logarithms, their uſes in arithmetic, and the manner in which their firſt inventors computed them, with a variety of other curious particulars concerning them. It is divided into a great number of ſeparate heads, with titles of their contents prefixed to them; and to theſe titles I refer the reader for a more particular account of what he may expect to find in it. As all the parts of this diſſertation are not of equal importance towards obtaining a

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Art.

juſt idea of logarithms, a direction is given to the reader in art. 266. what to attend to with moſt care, and what to paſs over more lightly.

I am ſenſible the preſent work is not without its imperfections: but theſe, I hope, the reader will excuſe, if upon the whole he finds it uſeful.

ELEMENTS OF PLANE TRIGONOMETRY.

ARTICLE I.

THE buſineſs of Trigonometry is, as its name imports, the menſuration of triangles, and conſiſts in diſcovering the angles of a triangle when the ſides are given, and the ſides, or proportions of the ſides, when the angles are given, and both ſides and angles when other ſides and angles are given: in order to which, it is neceſſary that not only the circumferences of circles, but alſo certain right lines in and about circles be ſuppoſed to be divided into ſome determinate number of parts.

Art. 2. Upon this account the antient Mathematicians thought fit to divide the periphery of a circle into 360 equal parts, which are called Degrees; and every degree into 60 parts, called Minutes, or Firſt Minutes; and every minute into 60 Seconds, or Second Minutes; and again, every ſecond into 60 Third Minutes, or Thirds; and ſo on. Every angle is ſaid to be of ſuch a number of degrees and minutes as there are degrees and minutes in the arc meaſuring that angle.

Art. 3. There are ſome that would have a degree divided into Centeſimal parts, rather than Sexageſimal ones; and it would perhaps be more uſeful to divide not only a degree, but even the whole circle, in a decuple ratio; which diviſion, 'tis not unlikely may ſome time or other take place. The advantage that would accrue from dividing the circle in a decuple proportion ariſes chiefly from the nature of the Notation now in uſe, by which the value of any number is augmented or diminiſhed in a decuple and ſubdecuple proportion, to which therefore ſuch a diviſion would be peculiarly adapted. But as the antients made uſe of a different Notation, this could be no advantage to them; they therefore on other accounts choſe the number 360: probably becauſe the year was thought by the *Egyptians*, to whom this diviſion of the circle is uſually aſcribed, to conſiſt of 360 days, or the circle which the Sun ſeemed to deſcribe in a year to be already diſtinguiſhed by nature into 360 parts. There was this farther convenience in this diviſion; that the number 360 is diviſible by ſeveral other numbers without a remainder; as by 10, 6, 4, 2, 90, 60, 9, *&c.* by which means they could accurately expreſs ſeveral parts of the periphery in whole numbers, which they eſteemed far clearer and leſs intricate than mixt numbers or fractions.

Art. 4. Now if the circumference of a circle contains 360 degrees, a quadrant or fourth part thereof will contain 90 degrees. And as a quadrant ſubtends a right angle at the center of the circle, a right angle is ſaid to be an angle of 90 Degrees. If the circumference had been divided into 100 parts, a Quadrant would have contained 25 of thoſe parts.

The Chord or *Subtenſe* of an Arc is a right line drawn from one end of the arc to the other.

The *Right Sine* of any arc, which is generally called only its *Sine*, is a perpendicular line, let fall from one extremity of that arc to a Diameter drawn through the other extremity.

The Sine divides the Diameter to which it is perpendicular into two parts, each of which is uſually called the Verſed Sine of that arc between which and the Right Sine it is intercepted. It is ſometimes term'd the Sagitta of that arc, or more properly of double that arc, which is compared to a bow having this line for its arrow.

'Tis evident from theſe definitions, that any arc DBO (*Fig* 1.) has the ſame Chord as DAO its difference from a whole circle; and any arc DB the ſame Sine as DHA its difference from a ſemicircle; which difference I ſhall henceforwards generally call its ſupplement to a ſemicircle, to diſtinguiſh it from the arc DH, or difference of DB from a Quadrant, which is uſually called its Complement to a Quadrant, or ſimply its Complement.

But DB and its ſupplement DHA have different Verſed Sines, EB and AE; the former of which, EB, belonging to an arc DB leſs than a quadrant, is equal to the difference of the radius and the line CE, and the latter to their Sum. The line CE is called the *Coſine* of either of thoſe arcs; for the definition of a *Coſine* is, that it is the Sine of the Complement of any arc to a quadrant, or the Sine of ſuch an arc as being added to, or taken from, the propoſed arc, ſhall make it equal to a quadrant: whence (in *Fig.* 1.) DF, or its equal EC (*El.* 1. 34,) is the Coſine of DB, or DA. The greateſt Sine poſſible is evidently equal to the Radius, which is therefore ſometimes called the *Whole Sine*.

If thro' one extremity D of the arc DB, or DA be drawn a right line from the center of the circle, and produc'd till it meet BG, or AK, (a line drawn touching the circle in the other extremity B, or A) in G or K; the line CG is called the *Secant*, and BG the *Tangent* of the arc DB; and CK and AK the Secant and Tangent of the arc DA. 'Tis evident from the ſimilarity and equality of the triangles ACK, BCG, that the ſecants CG, CK, and the tangents BG, AK, of the arc DB, and its ſupplement DA to a Semicircle, are equal, as the Sines were ſhewn to be before.

The *Coſecant* and *Cotangent* of any arc is the Secant and Tangent of the Complement of that arc; or of an arc which being added to, or taken from, the propoſed arc, ſhall make it equal to a Quadrant. Thus CI is the Coſecant of either of the arcs DB, DA, and HI their Cotangent.

Art. 5. From the definition of a Sine we may eaſily deduce the two following corollaries.

Corol. 1. The Sine of any arc is equal to half the chord of double that arc. For (by *El.* 3. 3.) the radius CEB being perpendicular to DO, will biſect it. Therefore DE (the Sine or the arc DB, or DA) is = one half of DO, (the chord of DBO, which is double of DB.)

Corol. 2. Hence the Sine of thirty degrees is equal to half the Radius. For it is equal to half the chord of ſixty degrees; which chord (becauſe the arch it ſubtends, to wit, 60 Degrees, is a ſixth part of the whole circumference, or 360 degrees) is the ſide of a regular Hexagon inſcrib'd in the circle, and is therefore equal to the Radius by El. 4. 15.

Art. 6. A Trigonometrical *Canon*, or Table, is a table expreſſing in numbers the lengths that the Sines, Tangents, and Secants of all the arcs in a quadrant, beginning from ſome ſmall arc, as a minute, and taken in a regular Arithmetical progreſſion, of which that ſmall arc is the common difference, have with reſpect to the Radius.

The expreſſion for the Radius is quite arbitrary; it may be called any number whatever, if the other lines are repreſented oy proportionable numbers. But it has been found moſt convenient in calculation to repreſent the Radius by ſome number, wherein a unit is the only ſignificant figure, as 1, 10000, 10,000,000. If it be repreſented by 10,000,000, or be conceiv'd to be divided into 10,000,000 equal parts, the numbers repreſenting the Sines, Tangents, and Secants, will expreſs how many ſuch parts of the radius, namely, ten-millionth parts, are contained in the lines they repreſent reſpectively. In *Sherwin's* tables the radius is called 10,000, altho' the calculations are carried to ſeven places of figures; the meaning of which is this: the radius is conceived to be divided into 10,000 equal parts; and one of theſe parts is made the unit or ſtandard, to which all the lines about the circle are referr'd, or of which they are conſider'd as parts or multiples. In the inſtances ſubjoin'd to the following propoſitions the radius is called 1, or is ſuppoſed not to be divided at all; but is itſelf made the unit or ſtandard of which all the other lines are conſider'd as either parts or multiples. 'Tis evident that on theſe ſuppoſitions, as long as 1 is the only ſignificant figure in the number repreſenting the radius, the ſignificant figures of all the other numbers will be the ſame; only their places will be integral or decimal, higher or lower, according as the unit is a leſſer or greater part of the radius.

Art. 7. By the help of ſuch a table, the Sine, Tangent, or Secant of a given arc or angle, may be known immediately. And *vice verſâ* we may find the arc or angle, to which a given Tangent, Sine, or Secant, belongs.

Note, In Trigonometry, when an arc is ſaid to be known or given, all that is meant is, that its proportion to the circumference is known or given, without any the leaſt regard to its real and abſolute length. And in like manner, a Sine, Secant or Tangent is ſaid to be known, when its proportion to the Radius is known. Thus, if the arc DB is known to be 30 degrees, it is ſaid to be known without any regard to the number of feet or inches it contains; and the Sine of 30 degrees is ſaid to be known, becauſe we know (by *Corol.* 2.) that it bears to the radius the proportion of $\frac{1}{2}$ to 1, without conſidering how many feet long the Radius, is; for whatever is the length of the Radius, the Sine of 30 Degrees will be half of it; and univerſally, any Sine, Tangent, or Secant in one Circle is to the Radius of that Circle, as the Sine, Tangent, or Secant of the ſame angle in another circle is to the Radius of that other circle; and alternately, any Sine, Tangent, or Secant, in one circle is to the Sine, Tangent, or Secant of the ſame angle in another circle, as the radius of the former to the radius of the latter.

N. B. A Degree is mark'd thus, °; a Minute thus, ′; a Second thus, ″; a Third thus ‴; and ſo on: for example; 30 degrees, 17 minutes, 40 ſeconds, and 33 thirds, is wrote thus; 30°, 17′, 40″, 33‴, the number of ſtrokes being always the number of diſtances between the arch over which it is wrote and a degree, or the number of times a degree muſt be divided by 60 to arrive at that minute of which the arch is a multiple: for if wrote at length, they would ſtand thus; $\frac{30}{1}+\frac{17}{60}\times\frac{40}{60\times60}+\frac{33}{60\times60\times60}$ of a degree. In the following propoſitions, R, ſignifies the Radius, S, a Sine, CoS, a Coſine, T, a Tangent, CoT, a Cotangent, and Sec, a Secant.

Art. 8. *The Conſtruction of the Trigonometrical Canon.*

THERE are but few lines in the circle, either Sines, Tangents, or Secants, whoſe ratios to the radius are known without calculation; nevertheleſs, 'tis from them that all the reſt muſt be diſcovered. To ſhew how this may be done, and to do it in a few examples, is the deſign of the following propoſitions.

PROP. I. *A Theorem.*

Any two ſides of a right-angled triangle being given, the other may be found by calculation.

For (*El.* 1. 47.) the ſquare of AC (*Fig* 2.) is equal to the ſum of the ſquares of AB, BC. Therefore, if AB, BC, are known, AC may be found by ſquaring them, adding their two ſquares together, and extracting the ſquare root of the ſum. Again, the ſquare of AB is equal to the difference of the ſquares of AC, and BC; therefore, when AC, BC, are known, AB will be found by ſquaring AC, and BC, ſubtracting BCq from ACq, and extracting the ſquare root of the remainder; and, if AC and AB are known, BC may be found in the ſame manner. QED.

Art. 9. PROP. II. *Problem.*

The Sine DE of the Arc DB and the Radius CD being given, to find the Coſine DF. (Fig. 1.)

Since CD and DE are given in the right-angled triangle CDE, the ſide CE, which is equal to DF the Coſine, will (by the laſt Prop.) be $= \sqrt{CDq - DEq}$.

Art.

Art. 10. *Example.*

Let DB be an arc of 30°, and the radius CD = 1, then will DE = .5, or .5000000, and DEq = .2500000, which ſubtracted from CDq, or 1, leaves CEq = .75000000000000, the ſquare root of which is .8660254 = the Coſ. 30°.

The operation is thus.

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.75000000000000(.8660254=Coſ. 36°.
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346404

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8660225

17323504).69937500
69282016

..655484

11. This Coſine is equal to $\frac{\surd 3}{2}$; for (*Fig.* 1.) CEq = CDq — DEq = 1 — $\frac{1}{4}$ = $\frac{3}{4}$; whence CE = $\surd\frac{3}{4}$ = $\frac{\surd 3}{\surd 4}$, or $\frac{\surd 3}{2}$. Therefore if the preceding operation is right, $\frac{\surd 3}{2}$ ought to be = .8660254;

and ſo we find it is: for $\sqrt{3}$ is $= 1.7320508$ (as will appear to any one who will take the trouble of extracting it), which divided by 2, gives the number .8660254, the ſame as was found before.

12. Corol. 1. The Sine of any arc being given, the tangent may be found by this proportion. CoS : R :: S : T. For from the Similarity of the triangles DCE, GCB, (*Fig.* 1.) we have CE : CB :: DE : BG.

Example.

Let it be required to find the tangent of 30°. DE the Sine is $\frac{1}{2}$, and CE the Coſine has been found to be $= \frac{\sqrt{3}}{2}$. Therefore as $\frac{\sqrt{3}}{2} : 1 :: \frac{1}{2} : \frac{1 \times \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} =$ tang. 30°. Therefore T, 30° = .5773502.

The Work is thus:

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1.7320508)1.00000000(.5773502+
          86602540
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          133974600
          121243556
          ---------
          .127310440
           121243556
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          ..60668840
            51961524
            --------
            .87073160
             86602540
             --------
            ..47062000
              34641016
              --------
              12420984
```

13. **Corol.** 2. The Sine of any arc being given, the Secant may be had by this Proportion; CoS, : R : : R : Secant. For CE : CB : : CD : CG.

Example.

Let it be required to find the Secant of 30°. CE the Coſine is = .8660254. Say therefore, as .8660254 : 1 : : 1 : Secant of 30°, = 1.1547005.

The Work is thus:

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.8660254)1.0000000(1.1547005
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          .13397460
            8660254
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           .47372060
            43301270
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            .40707900
             34641016
            ---------
             .60668840
              60621778
             ---------
               ...47062000
                  43301270
                  --------
                  .3760730
```

14. Corol. 3. The Sine of any arc being given, the Cotangent and Coſecant may be found by theſe Proportions; T : R : : R : CoT. and S : R : : R : Coſecant. For the triangles CHI, CBG, and CED, are ſimilar, becauſe the angles at H, B, and E are right angles, and the angles HIC, GCB, are equal, being the alternate angles made by the line CI croſſing the parallels HI, CB. Therefore GB : CB : : HC : HI, that is, T : R : : R : coT. and DE : CD : : CH : CI, that is, S : R : : R : Coſecant.

Example.

Let it be required to find the Cotangent and Coſecant of 30°. T was before found $= \frac{1}{\sqrt{3}}$. Say therefore as $\frac{1}{\sqrt{3}} : 1 :: 1 : \sqrt{3} = 1.7320508 =$ coT, 30°. $S = \frac{1}{2}$; ſay therefore, as $\frac{1}{2} : 1 :: 1 : 2 =$ coſecant of 30°.

15. From this corollary it follows, that the tangent of 45° is equal to the radius. For ſince the tangent and cotangent of 45° are equal (the angles of which they are the tangents being equal) the Radius, which is always a mean proportional between them, muſt in this caſe be equal to either of them. Or it will follow from *Corol.* 1. For now the ſine and coſine are equal; conſequently the Radius and tangent muſt be ſo too. *El.* 5. 14.

Or it may be proved thus. Since CBG is a right angle, BCG, and BGC muſt together be equal to a right angle, (*El.* 1. 32, *Corol.* 3.): therefore if BCG is an angle of 45 degrees, or half a right one, BGC muſt likewiſe be half a right one. Whence (*El.* 1. 6.) BG = BC.

16. It may ſometimes be convenient to ſee ſome of the principal ſines and tangents expreſſed in the form of whole numbers or vulgar fractions. Thoſe therefore that have been here calculated are ſet down in order thus. R = 1. $S, 30° = \frac{1}{2}$. $T, 30° = \frac{1}{\sqrt{3}}$ or $\sqrt{\frac{1}{3}}$. CoS. 30°, or $S, 60° = \sqrt{\frac{3}{4}}$

or $\frac{\surd 3}{2}$. CoT, 30°, or T, 60° $= \surd 3$. CoSec. 30°, or Secant of 60° $= 2$, or double the Radius. T, 45° $= 1$. And ſince the Secant of any angle is always $= \surd\overline{T^2 + R^2}$, we may add that the Sec. 45° $= \surd 2$.

17. Thus it appears, that when once the Sine of any angle is known, the Coſine, Tangent, Secant, Cotangent, and Coſecant may be derived from it by calculation; as may alſo the Verſed Sine, it being always equal to the Sum or difference of the Radius and coſine. All that remains therefore is to find out ſome method of calculating the Sines.

PROP. 3. *Problem.*

18. The Sine DE of any arc DB (*Fig.* 3.) being given, to find DM or BM the Sine of half that arc.

Rule.

Find the coſine of the given arc by the laſt propoſition, and ſubtract it from the Radius; the remainder will be the Verſed Sine; the ſquare of which, added to the ſquare of the given ſine, will be equal to the ſquare of the Chord of the given arc, and the ſquare root to the chord itſelf, the half of which is the ſine ſought. *Art.* 5.

For when DE is given, CE may be found by the laſt; and conſequently EB, which is the difference between the Coſine CE and the Radius. Therefore in the right-angled triangle DBE the two ſides DE and EB are given, and conſequently the remaining ſide BD may be found (by *Prop.* 1.); the half of which DM or BM, is the Sine of the arc DL, or LB, or half the given arc. QEI.

Example.

Let DE be the ſine of 30°; and let it be required to find DM the ſine of 15°. CE was found, by *Prop.* 2, to be = .8660254.

Therefore from 1.0000000 = CB
take .8660254 = CE
and there will remain .1339746 = EB; therefore EBq = .01794919344516. But becauſe DE = $\frac{1}{2}$, DEq will be = $\frac{1}{4}$ = .25, and therefore DEq + EBq or BDq = .26794919344516, the ſquare root of which is DB = .5176380, whereof the half .2588190 is = DM = S. 15°.

The Work is thus:

```
                                  .1339746 = EB
                                  .1339746 = EB
                                 ------------
                                  8038476
                                 5358984
                                9378222
                              12057714
                              4019238
                             4019238
                            1339746
                           --------------------
                           .01794919344516 = EBq
                           .25 = DEq
                           --------------------
DEq + EBq = BDq = . 26794919344516 (.5176380
                    25                  = BD
                   -----
          101) . 179
                 101
                 -----
         1027)7849
              7189
             ------
       10346).66019
              62076
              ------
      103523) .394334
               310569
              -------
    1035268) . 8376545
               8282144
              --------
   10352760) . . 9440116
                 0000000
                --------
                 9440116
```

PROP.

PROP. 4. *Problem.*

19. The Sine BM (*Fig.* 3.) of any arc being given, to find the Sine of double that arc.

Rule.

Find the Coſine CM of the given arc BL, by *Prop.* 2. and ſay; as Radius to Coſine, ſo is twice the given Sine to the Sine ſought.

For the triangles CBM, DBE, are equiangular, the angles at E and M being right angles, and the angle at B common to them both. Therefore (*El.* 6, 4,) we have this proportion, CB : CM : : DB, or 2 BM : DE. or R : CoS : : 2S : DE. QEI.

20. COROL. 1. Hence, when the Coſine of half any propoſed arc, and the ſubtenſe of the arc itſelf, are known, the chord of double that arc may be found by this proportion; as Radius to twice the Coſine of half the arc, ſo is the chord of the whole arc to the chord of double the arc. For let DB be the arc propoſed; the chord is DB; and CM is the Coſine of the arc DL = ½ DB arc. By the propoſition we had CB : CM : : DB : DE; therefore, doubling the conſequents, we have CB : 2CM : : DB : 2DE = DO, chord of DO, or 2DB.

21. COROL. 2. Hence, when the Sine of any arc, and the Sine of its double are known, the Coſine of the ſimple arc may be found by this proportion; S, ſimple arc : S, double arc : : ½

Rad, : CoS, ſimple arc. For let BL be the ſimple arc; DB will be the double arc: and BM, DE, are the ſines of BL, and DB, and are ſuppoſed to be known. By the laſt *Corol.* we had CB : 2CM : : DB, or 2BM : 2DE. Whence CB : 2CM : : BM : DE, and BM : DE : : $\frac{CB}{2}$: CM.

22. Corol. 3. Twice the rectangle under the Sine and coſine of any arc BL is equal to the rectangle under the Radius and Sine of a double arc. For in *Corol.* 2, we had CB : 2 CM : : BM : DE, whence (*El.* 6. 16.) 2CM × BM = CB × DE.

Example to Corol. 2.

23. Let BL be an arc of 15°. Its Sine BM was found in *Prop.* 3. to be = .2588190. DE the Sine of its double or 30°, is = $\frac{1}{2}$. It is required to find the coſine of 15°. Say, as .2588190 : $\frac{1}{2}$: : : $\frac{1}{2}$ (= $\frac{1}{2}$ Rad.) : $\frac{1}{4 \times .2588190}$ = $\frac{1}{1.0352760}$ = .9659259 = CoS, 15°.

The Work is thus.

```
1.0352760)1.00000000(.9659259
           93174840
           --------
         ..68251600
           62416560
           --------
           .61350400
            51763800
            --------
            .95866000
             93174840
             --------
             .26911600
              20705520
              --------
              .62060800
               51763800
               --------
               102970000
                93174840
                --------
                .9795160
```

P R O P. V. *A Problem.*

The Sines FO, and DK, (*Fig.* 4.) of any two arcs FD, BD, being given, to find FI the Sine of their ſum, and EL the Sine of their difference.

Conſtr.

Through the point O draw OP parallel to DK; and, producing FO till it meets the circumference in E, draw through the points O, E, the lines OM, EG parallel to CB, and meeting FI in M and G.

Dem.

Since the Sines FO, DK are given, the Coſines CO, CK, may be found by *Prop.* 2. And becauſe the triangles CDK, COP, CHI, FOH, FOM, are equiangular, the Sides about the equal angles will be proportional (*El.* 6. 4.); therefore CD : DK :: CO : OP; which will thence be known: And again, CD : CK :: FO : FM, which will thence be known likewiſe. But OP = MI (*El.* 1. 34.) Therefore OP + FM is = MI + FM = FI, the Sine of FB the ſum of the arcs. And ſince OM is parallel to EG the baſe of the triangle FGE, we ſhall have (by *El.* 6. 2.) FO : OE :: FM : MG. But FO = OE, by *Art.* 5. Therefore FM = MG = (*El.* 1. 34.) ON. Whence OP — FM is = OP — ON = NP = (*El.* 1. 34.) EL, the Sine of EB the difference of the arcs. QEI.

25. Coroll. It may be obſerved that, as the differences of the arcs BE, BD, BF, are equal, the arc BD is an arithmetical mean between BE and BF; or in other words, If two unequal arcs (or any other two unequal quantities) be added together, the greater arc will be an Arithmetical Mean between the Sum and difference of the arcs.

Example.

26. By way of example to the firſt part of this propoſition, let it be required to find the

Sine of 45 degrees, or of the ſum of 30° and 15°; having given the ſines of 30° and 15°.

DB arc = 30°. FD = 15°. DK = ½. FO = .2588190. CO = .9659259 CK = .8660254. CO & CK are not given, but (as was before obſerved) are to be found by the help of *Prop.* 2. Having found them to be the numbers here ſet down, we ſay, as 1 : ½ : : .9659259 : .4829629 = OP; and as 1 : .8660254 : : .2588190 :

.2241438 = FM
.4829629 = OP

.7071067 = FI = S, 45°.

The Work is thus.

$$\begin{array}{r}
.2588190 \\
.8660254 \\
\hline
10352760 \\
12940950 \\
5176380 \\
155291400 \\
15529140 \\
20705520 \\
\hline
.22414382800260
\end{array}$$

PROP. 6.

27. If three Arcs are in Arithmetical proportion, the Radius will be to double the Coſine of the middle arc, as the Sine of the common difference of the arcs is to the difference of the Sines of the extreme arcs.

For, (*Fig.* 4.) let BE, BD, BF, be three arcs in Arithmetical proportion; their common difference is the arc FD, and its Sine is FO. FG is the difference of FI, and EL, the Sines of the extremes. It was proved, in *Prop.* 5, that CD : CK :: FO : FM; wherefore, doubling the Conſequents, CD : 2 CK :: FO : 2 FM = FG. QED.

28. Corol. 1. If the middle arc BD is an arc of 60°, the Sine of the common difference of the arcs will be equal to the difference of the Sines of extreme arcs. For in that Caſe, CK the Coſine of BD will be S, 30°, the double whereof is equal to the radius, or CD. ConſequentlyFO, FG are equal.

29. Corol. 2. Hence if the Sines of all arcs that are diſtant from one another by a given interval, or that are in Arithmetical proportion, from the beginning of the quadrant to 60° are given, the other Sines may be found by one addition only; namely, of the Sine of the common difference to the Sine of the leſſer extreme. Thus, the Sine of 61° is = S, 59° + S, 1°; and the Sine of 62° = S, 58° + S, 2°. Alſo, S, 63° = S, 57° + S, 3°; and ſo on. And in the ſame manner, if the Sines have been calculated to every minute of the Quadrant from 0 up to 60°, the remaining ſines may be found alſo to every minute from 60° to 90°. For the Sine of 60°, 1′, is = S, 59°, 59′, + S, 1′; and S, 60°, 2′ = S, 59°, 58′ + S, 2′, and ſo of the reſt.

How to calculate the Sines of Arcs in Arithmetical progreſſion from the beginning of the Quadrant to 60°, having given the Sine of 1 minute, will appear from

30. Corol. 3. If the Sines of all arcs from the beginning of the quadrant to any part of it,

that differ from one another by a given difference, are given, we may thence find the Sines of all the arcs to the double of that part. For example, if all the Sines from the beginning of the quadrant to 15° are given to every degree of the quadrant, then, by the proportion laid down in this propoſition, we may find all the Sines to 30° to every degree of the quadrant. For Radius is to double the coſine of 15°, as the Sine of 1 degree to the difference of the Sines of 14° and 16°, theſe arcs 14°, 15°, 16°, being in Arithmetical proportion; for the ſame reaſon, R : 2 CoS. 15° : : S, 2° : diff. of S, 13°, and S, 17°, and again R : 2 CoS. 15° : : S, 3° : diff. S, 12° and S, 18°. Whence by adding the differences of the Sines of 14 and 16 degrees, 13 and 17 degrees, and 12 and 18 degrees, to the Sines of 14°, 13°, and 12° reſpectively, we get the Sines of 16°, 17°, and 18°. And in this manner may all the ſines for every degree from 15 to 30 degrees be calculated; for till we come to 30°, 15° may be an arithmetical mean between the arc whoſe Sine is ſought, and ſome one of the arcs under 15°, whoſe Sines are known.

But when we have found all the Sines to 30°, we muſt make uſe of a new proportion, and ſay, as Radius to double the coſine of 30°, ſo is the Sine of 1 degree to the difference of the Sines of 29°, and 31°; and again, as R : 2 CoS. 30° : : S, 2° : diff. S, 28 and S, 32°; and R : 2 CoS, 30° : : S, 3° : diff. S, 27° and S, 33°; and ſo on, till we come to 60 degrees.

In this caſe we need only multiply the Sines of the common differences by $\sqrt{3} = 1.7320508075688$

688773 in every operation, and the products will be the differences of the Sines of the extremes reſpectively. For in the Example to *Prop.* 2. it appeared that the CoS. 30° was $=\frac{\sqrt{3}}{2}$; wherefore double that Coſine is $=\sqrt{3}$. And the diviſion by the radius or 1 has no effect, and muſt be omitted.

In the ſame manner if the Sines from 0° to 15° and from 15 to 30° are known to every Minute of the quadrant (inſtead of every degree), the Sines from 15° to 30°, and from 30° to 60° may be found to every minute of the Quadrant; as was obſerved in the laſt corollary.

And by this Method, if only the Sine of 1 Minute be given, we may calculate all the Sines from 1′ to 15° (and thence from 15° to 30°, and from 30° to 60°) to every minute of the quadrant. For having S, 1, the CoS, 1′ may be found by *Prop.* 2; and by *Prop.* 4, the Sine of 2′, the coſine of which may be found by *Prop.* 2; ſo that we may now find the Sine of 3 minutes in the ſame manner as before, by ſaying, as R : 2 CoS. 2′ : : S, 1′ : diff. S, 1′ and S, 3′. And again, as R : 2 CoS. 2′ : : S, 2′ : diff. S, 0′ and S, 4′ = S, 4′. And (finding the CoS. 4′ by *Prop.* 2.) R : 2 CoS, 4′ : : S, 1′ : diff. S, 3′ and S, 5′. And again, as R : 2 CoS. 4′ : : S, 2′ : diff. S, 2′ and S, 6′; and R : 2 CoS. 4′ : : S, 3′ : diff. S, 1′ and S, 7′; and R : 2 CoS. 4′ : : S, 4′ : diff. S, 0′ and S, 8′ = S, 8′. And ſo for the reſt.

31. LEMMA 1. Any two lines AO, FO, drawn from the extremities A, F, of a curve AF, all

the parts of which are concave the ſame way, and both falling on the convex ſide of the curve-line, are greater than the curve.

Join the points A, F, (*Fig.* 5.) and biſecting AF in R, draw RV perpendicular to it, meeting the curve in V. From any other point, as H, in the curve, draw HI parallel to AF, and meeting the curve in I; and divide the arcs AI, HF, into any number of parts AM, MI, FG, GH, and draw the chords of theſe parts. Through the points M, I, H, G, draw KB, LC, PD, TE, at right angles to AF, meeting the lines AO, FO, in K, L, P, T, and AF in B,C,D,E. Through M and G draw MX, GY parallel to AF; and MN, GS, parallel to AO, FO reſpectively. Laſtly, join L, P, and draw LZ parallel to AF.

Since ABM is a right angle, AMB muſt be acute and AMK obtuſe. Therefore AK is greater than AM (*El.* 1. 19.) Again, MXI being a right angle, MIX muſt be acute, and MIN obtuſe; wherefore MN, or its equal KL (*El.* 1. 34.) is greater than MI. And LP is greater than LZ or IH; conſequently LO + OP is much greater than IH. In like manner it may be ſhewn that PT is greater than HG, and TF than GF. Therefore AO + OF is greater than the Sum of all the chords; that is, when the number of chords is infinite, and every chord is equal to its arch (for the demonſtration does not depend upon any particular number of chords, and therefore will hold good when the number is infinite) than the curve AF. QED.

This proof may be thrown into the form of an *ex abſurdo* demonſtration, aſſuming as a poſtulatum that an arc may be divided into ſo many parts that it's difference from the ſum of all the chords ſhall be leſs than any finite quantity whatever. But it's evidence would ſcarce be made greater than it is already.

32. Corol. Hence the tangent of any arc in a circle is greater than the arc itſelf. For let the curve AF (*Fig.* 6.) be a circle whoſe center is *a*, and let AO, OF be tangents to it in the points A, F. Join *a*, O, and draw the radii *a*A, *a*F. By *El.* 3. 18, the angles *a*AO, *a*FO, are right ones; therefore *a*OF, and *a*OA are each of them leſs than a right. And 'tis evident F*a*, *a*O, are reſpectively equal, and therefore proportional, to A*a*, *a*O. Conſequently (*El.* 6. 7.) the angles FO*a*, AO*a* are equal, and likewiſe A*a*O, F*a*O. Therefore (*El.* 1. 4.) AO = FO. And the arcs AV, VF are equal (V being the point in which *a*O cuts AF) becauſe the angles A*a*O, F*a*O, were ſhewn to be ſo. Therefore, ſince by the propoſition AO + FO is greater than AF, it follows that FO, the tangent of FV, is greater than the arc FV.

PROP. 7. *Theorem.*

33. In very ſmall arcs the ſine and tangent of the ſame arc are to each other very nearly in a ratio of equality.

For from the Similarity of the triangles CED, CBG, we have CE : CB :: ED : BG. This is univerſal; but when the point D approaches to B, the line EB will become exceeding ſmall in compariſon to CE, or CB, which will therefore be very nearly equal to each other: conſequently the Sine ED will be very nearly equal to the tangent BG. To illuſtrate this farther, draw DF parallel to CB: the triangles GDF, GCB, are ſimilar; therefore DF : CB : : GF : GB. Whence if DF, or EB is leſs than $\frac{1}{10,000,000}$th part of the radius CB, GF the difference between the Sine and tangent will be leſs than the $\frac{1}{10,000,000}$th part of the tangent GB.

34. COROL. 1. Since the arc is greater than its Sine, and leſs than its tangent, and the Sine and tangent of a very ſmall arc are nearly equal, the arc which lies between them muſt be nearly equal to either of them. Wherefore in very ſmall arcs, we may ſubſtitute the proportion of the arcs inſtead of that of the ſines, and ſay, without erring much from the truth, as arc to arc, ſo is ſine to ſine.

35. COROL. 2. It may be obſerved here that the Chord, as well as the arc, approaches continually to an equality with either the Sine or tangent; and for the ſame reaſon, to wit, becauſe it lies between them. And when the arc is diminiſhed till it vaniſh, the ultimate ratio of the Sine, Chord, arc, and tangent is a ratio of equality.

That the Arc as it grows leſs is continually in a leſs and leſs proportion to its chord is evident likewiſe from this conſideration; ſince the chord is always greater than the Sine, the arc will always bear a leſs proportion to it than to the Sine; that is, a leſs proportion than a double arc bears to double the Sine of the ſingle arc, or (*Art.* 5.) to its own chord.

PROP. 8.

To find the Sine of an arc of 1 *Minute.*

36. By continually biſecting an arc of 30°, whoſe ſine is known to be $\frac{1}{2}$, and finding the ſines all the way by *Prop.* 3. we ſhall at laſt get the Sine of 52″, 44‴, 3⁗, 45′′′′′; the coſine of which is very nearly equal to the Radius; in which caſe (as appears by the laſt *Propoſition)* the Sines may be eſteemed proportional to the Arcs. Say therefore, as an arc of 52″, 44‴, 3⁗, 45′′′′′, is to an arc of 1′, ſo is the Sine of the former arc to the Sine of 1 minute. When the Sine of 1 minute is thus found, the Sine and coſine of 2 minutes may be found by *Prop.* 4. and 2.

Note, Whoever would find the Sine of 1 minute by this method, muſt make all his calculations to three or four places of decimals farther than he intends ſhould be accurate; becauſe the error, though ſmall at the beginning of the calculation, will be multiplied and encreaſed every new operation, ſo as without this caution to become conſiderable at laſt. Thus if the Sine of 52″, 44‴, 3⁗, 45′′′′′, is to be found accurately to 7 decimal places, the Sine of 15°, of 7°, 30′, and of all the reſt muſt be calculated to 10 or 11 decimal places.

The intermediate arcs are theſe that follow.

1ſt arc = $30^\circ . 00' \; 00'' . 00''' . 00'''' . 00'''''$

2d arc = 15° . . . .

34 = $7^\circ . 30'$

4th = $3^\circ . 4'$

5th = $1^\circ . 52' . 30''$

6th = . $56' . 15''$

7th = $28' . 7'' . 30'''$

8th = $14' . 3'' . 45'''$

9th = $7' . 1'' . 52''' . 30''''$

10th = $3' . 30'' . 56''' . 15''''$

11th = $1' . 45'' . 28''' . 7'''' . 30'''''$

12th = $. 52'' . 44''' . 3'''' . 45'''''$

PROP. 9. *Theorem.*

37. If any angle BAC ſituated in the circumference of a circle is biſected by a right line AD, and AC is produced till it is cut by DE which is made equal to AD; the external line CE will be equal to the chord AB.

In the quadrilateral Figure ABDC (*Fig.* 8.) inſcribed in the circle whoſe diameter is AF, the angles B, and ACD, are together equal to two right angles (*El.* 3. 22.) and conſequently to the two angles ACD, DCE (*El.* 1. 13.) Therefore B=DCE. But the angle E is = DAC (*El.* 1. 5.) becauſe DE is by conſtruction equal to DA; and DAC = DAB *ex hypotheſi.* Therefore E= DAB. And the arches CD, DB, and their chords are equal (*El.* 3. 29.). Therefore (*El.* 1. 26.) the triangles DAB, DCE are every way equal and alike, and CE = BA, QED.

PROP.

PROP. 10. *Theorem.*

38. If any number of arches AB, AC, AD, be in Arithmetical proportion, the chord of the firſt is to the chord of the Second, as the chord of the ſecond is to the Sum of the chords of the firſt and third, and as any one of the chords is to the Sum of the preceding and following chords.

Let the chords (*Fig.* 9.) be AB, AC, AD, AE, AF, AG. Produce AD to H, ſo that AC H may be an iſoſceles triangle; and in like manner AE to I, AF to K, AG to L, ſo that ADI, AEK, AFL, ſhall be iſoſceles triangles.

Then ſince the arch BC is equal to the arch CD, the angle BAD is biſected by AC; therefore DH = AB, by the laſt. For the ſame reaſon EI = AC; FK = AD, and GL = AE. But becauſe the angles at A ſubtended by the equal arches AB, BC, CD, DE, &c. are equal, the ſeveral iſoſceles triangles ABC, ACH, ADI, AEK, AFL, are equiangular, and conſequently ſimilar (*El.* 6. 4.) whence AB : AC :: AC : AH :: AD : AI :: AE : AK :: AF : AL; that is, AB : AC :: AC : AB + AD :: AD : AC + AE :: AE : AD + AF :: AF : AE + AG. QED.

39. COROL. Since (by *Prop.* 4. *Corol.* 1.) the Radius is to double the Coſine of $\frac{1}{2}$, AB, (*Fig.* 9.) as the chord AB to the chord AC, it follows that, R : 2 CoS. $\frac{1}{2}$ AB :: AB : AC :: AC : AB + AD :: AD : AC + AE, &c. or (halving theſe chords)

R :

$R : 2\,\mathrm{CoS.}\ \frac{1}{2}\,AB :: \frac{AB}{2} : \frac{AC}{2} :: \frac{AC}{2} : \frac{AB}{2} + \frac{AD}{2} :: \frac{AD}{2} : \frac{AC}{2} + \frac{AE}{2}$ &c.; that is, as Radius is to twice the coſine of half the firſt arch, ſo is the ſine of half the firſt arch to the Sine of the firſt or of half the ſecond; ſo the Sine of half the ſecond arch to the ſum of the Sines of half the firſt and half the third; ſo the ſine of half the third to the Sum of the Sines of half the ſecond and half the fourth; ſo the ſine of half the fourth to the Sum of the Sines of half the third and half the fifth; and in general, ſo the ſine of half any one of the arcs to the ſum of the Sines of half the preceding and half the following arches.

40. Let the arch AB (*Fig.* 9) be 2 Minutes; AC 4 Minutes; AD 6′; AE 8′; AF 10′; AG 12′. Then the Sine and coſine of half the firſt arch or of ½ AB, will be the Sine and Coſine of 1 Minute; the Sine of half the ſecond arch AC will be the Sine of 2′; the Sine of ½ AD arch will be = S. 3′; the Sine of ½ AE arch will be = S, 4′; S, ½ AF arch = S, 5′; S, ½ AG arch = S, 6′. Whence having found the Sine of 1′ by *P.* 8, and its coſine by *P.* 2, the S, 2′ is found by ſaying, as R : 2 CoS, 1′ :: S, 1′ : S, 2′. And from theſe the Sines of all the following arcs are found with great eaſe to every minute of the quadrant, as follows. Call the Coſine of 1′ or the Sine of 89°, 59′, Q, and make the following proportions. $R : 2Q :: S,\ 2' \left(= \frac{AC}{2}\right) : S,\ 1' + S,\ 3' \left(= \frac{AB}{2} + \frac{AD}{2}\right)$, whence the Sine of 3′ is known. Again, R : 2Q :: S, 3′

AD

$\left(=\frac{AD}{2}\right)$ S, 2′, + S, 4′$\left(=\frac{AC}{2}+\frac{AE}{2}\right)$, whence the S, 4′ is known. Again, R : 2Q :: S, 4′, $\left(=\frac{AE}{2}\right)$: S, 3′ + S, 5′$\left(=\frac{AD}{2}+\frac{AF}{2}\right)$, whence the S, 5′ is known. Again, R : 2Q :: S, 5′, $\left(=\frac{AF}{2}\right)$: S, 4′ + S, 6′$\left(=\frac{AE}{2}+\frac{AG}{2}\right)$, whence the S, 6′ is known.

And in this manner may all the Sines be calculated to every minute of the Quadrant. And as the Radius or firſt term of the proportion, is a unit, and conſequently a diviſion by it has no effect, the operations may be performed with great eaſe by a multiplication only (which may be contracted according to *Oughtred's* rule) and a ſubtraction.

41. *A Short review of the Method of making Sines.*

As it is evident that the preceding propoſitions are not all of equal uſe and importance in calculating a table of Sines, but ſome are only ſubſervient to the demonſtrations of others of more immediate conſequence; it cannot but be uſeful to take a diſtinct view of their ſeveral dependencies one upon another, and to inquire in what order and to what purpoſes a perſon that was calculating ſuch a table would apply them. And firſt he would by the help of the 3d *Propoſition* (which depends upon the Second, and that on the firſt) find the Sines of 15°; 7°, 30′, *&c.* till he came to $52'', 44''', 3'''', 45'''''$; and then by *P.* 8, (to which the *Lemma* and *P.* 7. are ſubſervient) he would find the Sine of 1′; which having obtained, he would by *P.* 10, *Corol.* (to which *P.* 9, and *P.*

2.

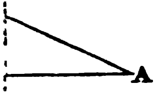
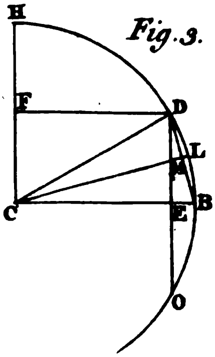


Fig. 3.



5.

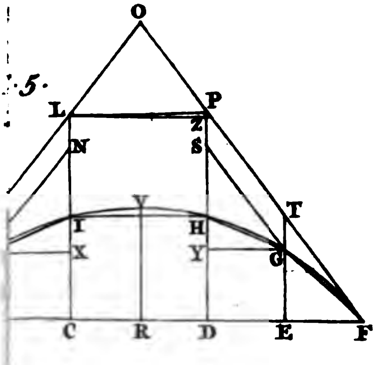
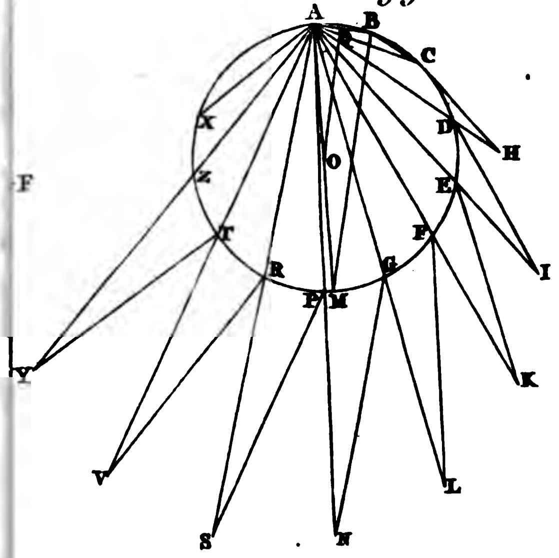


Fig. 9.



J. Mynde sc.

rol. 1. are neceſſary) compute all the Sines to 60°; after which inſtead of continuing to find the remaining Sines in the ſame manner, he would make uſe of *P.* 6. *Corol.* 2. (to which *P.* 5 is ſubſervient), becauſe by that Corollary the Sines from 60 to 90 degrees may be found by mere additions.

The ſines from the beginning of the quadrant to 60° might be computed by *P.* 6. *Corol.* 3; but with rather leſs eaſe than by *P.* 10. *Corol.* becauſe in the latter the ratio of the radius to twice the Coſine of 1′ ſerves throughout the whole calculation; whereas in the former it is neceſſary every time you have found all the Sines to double of any propos'd arc, to calculate a new coſine: as firſt the CoS. 2′, then the CoS. 4′, and ſo on.

Scholium.

42. That *great Geometrician*, and *incomparable Philoſopher*, Sir *Iſaac Newton*, was the firſt that invented infinite converging ſerieſes, by which when the arc was given the Sine might be computed. For if the radius be r, and any arc be called a, he found the Sine would be $a \frac{-a^3}{1.2.3.r.^2} + \frac{a^5}{1.2.3.4.5.r^4} \frac{-a^7}{1.2.3.4.5.6.7.r.^6} \frac{+a^9}{1.2.3.4.5.6.7.8.9.r^8}$ &c. and the coſine $= r \frac{-a^2}{1.2.r} \frac{+a^4}{1.2.3.4.r^3} \frac{-a^6}{1.2.3.4.5.6r^5} \frac{+a^8}{1.2.3.4.5.6.7.8.r^7} \frac{-a^{10}}{1.2.3.4.5.6.7.8.9.10.r^9}$ &c. the inveſtigation of

theſe Serieſes ſhall be given hereafter. At preſent we will obſerve that by the firſt of theſe Serieſes the Sine of 1 minute may be found with much greater eaſe than by *Prop.* 1; if the length of that arc be known. The length of an arc is ſaid to be known, when its proportion to the radius is known; to which purpoſe it is neceſſary that the proportion of the circumference of a circle to its diameter be known; for when an arc is ſaid to be of ſo many degrees or minutes, we only know its proportion to the circumferences or ſemicircumference, or how many 360th parts, or 21600th parts, of the circumference the arc is equal to; but to compare it with the radius, we muſt know the ratio of the circumference to the Diameter, that ſo by the mediation of the circumference we may find the proportion between the arc and diameter, or radius. Now the proportion of the Radius to the ſemicircumference, or the Diameter to the circumference, has been found by a Method invented by *Archimedes* (which Dr. *Saunderſon* has copiouſly and clearly explained in his *Elements of Algebra*, from *Art.* 340 to 347) to be that of 1 to 3.1415, 9265, 3589, 7932,38. An arch of 1 minute is the 60th part of the 180th part of the ſemicircumference, or the $\frac{1}{60 \times 180}$th part of 3,1415,9265,3589,7932,38 (the radius being 1); which is =

$$\frac{3.1415,9265,3589,793238}{10800} = .0002908882086657$$

(21

Here therefore $a = 0.000290888208665721$

$a^3 = 0{,}000000000024613782$

a^5

$\frac{a^3}{1.2.3r^2} = \frac{a^3}{1.2.3.} = \frac{a^3}{6} = .0000000000041022297.$

Whence S, 1′ = 0,000290888204563424. In tables which go but to 7 places it is called ,0002909.

Theſe Serieſes in the beginning of the quadrant, when the arc a is but ſmall, converge extremely faſt. For in the Series for the Sine (an inſtance of which we have juſt now given) if a does not exceed 10 minutes, the two firſt terms thereof, to wit, $a \frac{-a^3}{6}$ (the radius r being ſuppoſed = 1) give the Sine to 15 places of figures: and if the arc a does not exceed 1 degree, the three firſt terms will exhibit the Sine to 15 places of figures. So that theſe Serieſes are very uſeful for finding the firſt Sines and Coſines, or the firſt and laſt Sines, of the Quadrant. But the greater the arc a is, the greater is the number of terms required to expreſs the Sine to a given degree of exactneſs: and at laſt when the arc is nearly equal to or greater than the radius, the ſeries converges but ſlowly. To remedy this, Dr. *Keill* deviſed other Serieſes ſimilar to the *Newtonian* ones, and derived from them, wherein he ſuppoſes the arc whoſe ſine is ſought to be the ſum or difference of two arcs, or (calling thoſe arcs A and z) to be = A+z or A—z. Let the Sine of the arc A be called a, and the Coſine b, then will the Sine of the arc A+z be $= a \frac{+bz}{1} - \frac{az^2}{1.2.} - \frac{bz^3}{1.2.3.} + \frac{az^4}{1.2.3.4.} + \frac{bz^5}{1.2.3.4.5.} \frac{-az^6}{1.2.3.4.5.6.} \frac{-bz^7}{1.2.3.4.5.6.7.} +$, &c. and its coſine $= b \frac{-az}{1} -$

$\frac{-bz^2}{1.2.} + \frac{az^3}{1.2.3.} + \frac{bz^4}{1.2.3.4.}\ \frac{-az^5}{1.2.3.4.5.}$
$\frac{-bz^6}{1.2.3.4.5.6.} + \frac{az^7}{1.2.3.4.5.6.7.} + \frac{bz^8}{1.2.3.4.5.6.7.8.}$
$\frac{-az^9}{1.2.3.4.5.6.7.8.9.}\ \frac{-bz^{10}}{1.2.3.4.5.6.7.8.9.10.}$ &c. And in like manner the Sine of the arc A—z is a
$\frac{-bz}{1}\ \frac{-az^2}{1.2.}\ \frac{+bz^3}{1.2.3.}\ \frac{+az^4}{1.2.3.4.}\ \frac{-bz^5}{1.2.3.4.5.}\ \frac{-az^6}{1.2.3.4.5.6.}$
$\frac{+bz^7}{1.2.3.4.5.6.7.}\ \frac{+az^8}{1.2.3.4.5.6.7.8.}\ \frac{-bz^9}{1.2.3.4.5.6.7.8.9.}$
$\frac{-az^{10}}{1.2.3.4.5.6.7.8.9.10.}$, &c. and its Coſine is $=$
$b\frac{+az}{1}\ \frac{-bz^2}{1.2.}\ \frac{-az^3}{1.2.3.}\ \frac{+bz^4}{1.2.3.4.}\ \frac{+az^5}{1.2.3.4.5.}$
$\frac{-bz^6}{1.2.3.4.5.6.}\ \frac{-az^7}{1.2.3.4.5.6.7.}\ \frac{+bz^8}{1.2.3.4.5.6.7.8.}$
$\frac{+az^9}{1.2.3.4.5.6.7.8.9}$,&c. the difference of the Sines of A and A+z is $\frac{bz}{1.}\ \frac{-az^2}{1.2.}\ \frac{-bz^3}{1.2.3.}\ \frac{+az^4}{1.2.3.4.}\ \frac{+bz^5}{1.2.3.4.5.}$
$\frac{-az^6}{1.2.3.4.5.6.}\ \frac{-bz^7}{1.2.3.4.5.6.7.}$, &c: and the difference of the Sines of A and A—z is $= \frac{bz}{1}$
$\frac{+az^2}{1.2.}\ \frac{-bz^3}{1.2.3.}\ \frac{-az^4}{1.2.3.4.}\ \frac{+bz^5}{1.2.3.4.5.}\ \frac{+az^6}{1.2.3.4.5.6.}$ &c.
whence the difference of theſe differences, or the ſecond difference, will be $\frac{2az^2}{1.2.}\ \frac{-2az^4}{1.2.3.4.}\ \frac{+2az^6}{1.2.3.4.5.6.}$
$\frac{-2az}{1.2.3.4.5.6.7.8.}\ \frac{+2az^{10}}{1.2.3.4.5.6.7.8.9.10.}$, &c. or $2a$ ×

$\overline{\frac{z^2}{1.2.} \quad \frac{-z^4}{1.2.3.4.} \quad \frac{+z^6}{1.2.3.4.5.6.} \quad \frac{-z^8}{1.2.3.4.5.6.7.8.}}$

$\frac{+z^{10}}{1.2.3.4.5.6.7.8.9.10.}$, &c; which is the product of twice the Sine of the arc A multiplied into the verſed Sine of the arc z, and converges very ſoon: ſo that if z be a firſt minute, the firſt term of the Series gives the 2d difference to 15 places of figures, and the ſecond term to 25 places. From hence if the Sines of any two arcs whoſe difference is one minute be given, the Sines of all the arcs that are in the ſame progreſſion may be found by an exceeding eaſy operation.

In the firſt and ſecond ſeries if $A = 0$, then will $a = 0$, and b it's coſine will become radius or 1. and hence if the terms wherein a is concerned are dropt, and 1 is ſubſtituted inſtead of b, the serieſes will be converted into the *Newtonian* ones for the Sine and Coſine of the arc z.

In the third and fourth ſeries if $A = 90°$, we ſhall have $b = 0$, and $a = 1$. whence again, taking away all the terms wherein b is concerned, and putting 1 inſtead of a, we ſhall have the *Newtonian* ſeries for the Sine and coſine of the complement of z to a right angle, or for the coſine and Sine of z.

All theſe Serieſes eaſily flow from the *Newtonian* ones, by means of the fifth propoſition 5. as is ſhewn by Mr. *Ham*, in his Appendix to *Cunn*'s Euclid in the following manner.

Explanation

Explanation and derivation of the foregoing Serieſes of Dr. Keill.

Let the arc BD, (*fig.* 4.) which is ſuppoſed to be known, be called A; it's ſine DK, a; and coſine CK, b. call the arc DF, or DE, which is alſo ſuppoſed to be known, z. It is required to find FI, the Sine of A + z, or EL the Sine of A − z.

By the *Newtonian* Series the Sine FO is $= z \frac{-z^3}{1.2.3.} \frac{+z^5}{1.2.3.4.5.} \frac{-z^7}{1.2.3.4.5.6.7.}$, &c. and the Coſine CO $= 1 \frac{-z^2}{1.2.} \frac{+z^4}{1.2.3.4.} \frac{-z^6}{1.2.3.4.5.6.}$ &c. and becauſe CD : DK :: CO : OP, or 1 : a :: CO : OP, it follows that OP $= a \times$ CO $= a \frac{-az^2}{1.2.} \frac{+az^4}{1.2.3.4.} \frac{-az^6}{1.2.3.4.5.6.}$ &c. Again becauſe the triangles CDK, FOM, are ſimilar, we have CD : CK :: FO : FM, or 1 : b :: FO : FM; whence FM $= b \times$ FO $= \frac{bz}{1.} \frac{-bz^3}{1.2.3.} \frac{+bz^5}{1.2.3.4.5.} \frac{-bz^7}{1.2.3.4.5.6.7.}$ &c. Therefore FI (= OP + FM) $= a \frac{+bz}{1.} \frac{-az^2}{1.2.} \frac{-bz^3}{1.2.3.} \frac{+az^4}{1.2.3.4.} \frac{+bz^5}{1.2.3.4.5.} \frac{-az^6}{1.2.3.4.5.6.} \frac{-bz^7}{1.2.3.4.5.6.7.}$ &c; and EL (= OP −ON = OP − FM) $= a \frac{-bz}{1.} \frac{-az^2}{1.2.} \frac{+bz^3}{1.2.3.} \frac{+az^4}{1.2.3.4.} \frac{-bz^5}{1.2.3.4.5.} \frac{-az^6}{1.2.3.4.5.6.} \frac{+bz^7}{1.2.3.4.5.6.7.}$

+

$\frac{+az^8}{1.2.3.4.5.6.7.8.}$ &c.

Thus we have found the Sines of the arcs $A + z$ and $A - z$. To find their coſines CI and CL; we muſt get expreſſions for CP and MO or IP.

Since CD : CK :: CO : CP, or 1 : b :: CO : CP, it follows that CP $= b \times$ CO $= b \frac{-bz^2}{1.2.} \frac{+bz^4}{1.2.3.4.} \frac{-bz^6}{1.2.3.4.5.6} \frac{+bz^8}{1.2.3.4.5.6.7.8}$ &c: and becauſe the triangles CDK, FMO are ſimilar, we have CD : DK :: FO : MO, or 1 : a :: FO : MO; whence MO $= a \times$ FO $= az \frac{-az^3}{1.2.3.} \frac{+az^5}{1.2.3.4.5.} \frac{-az^7}{1.2.3.4.5.6.7.}$ &c. Therefore CI (= CP − IP = CP − MO) $= b \frac{-az}{1.} \frac{-bz^2}{1.2.} \frac{+az^3}{1.2.3.} \frac{+bz^4}{1.2.3.4.} \frac{-az^5}{1.2.3.4.5.} \frac{-bz^6}{1.2.3.4.5.6.}$ &c; and CL (= CP + PL = CP + NE = CP + MO) $= b \frac{+az}{1} \frac{-bz^2}{1.2.} \frac{-az^3}{1.2.3.} \frac{+bz^4}{1.2.3.4.} \frac{+az^5}{1.2.3.4.5.} \frac{-bz^6}{1.2.3.4.5.6.} \frac{-az^7}{1.2.3.4.5.6.7.}$ +, &c. Q.E.I.

The difference of the differences of the ſines of $A - z$, A, $A + z$, has been found $= 2a \times \overline{\frac{z^2}{1.2.} \frac{-z^4}{1.2.3.4.} \frac{+z^6}{1.2.3.4.5.6.} \frac{-z^8}{1.2.3.4.5.6.7.8.}}$ &c. or to the product of twice the Sine of the Mean arc A into the verſed ſine of z; for if the coſine of z, to wit, $1 \frac{-z^2}{1.2.} \frac{+z^4}{1.2.3.4.} \frac{-z^6}{1.2.3.4.5.6.}$ &c. be ſubtracted from the radius 1, or added to it with all the ſines changed, it will leave the Verſed Sine $= \frac{z^2}{1.2.} \frac{-z^4}{1.2.3.4.} \frac{+z^6}{1.2.3.4.5.6.}$ &c.

Now if this Second difference be ſubtracted from $2a$ or twice the Sine of the Mean arc A, the remainder will be $2a \frac{-2az^2}{1.2.} \frac{+2az^4}{1.2.3.4.} \frac{-2az^6}{1.2.3.4.5.6.} \frac{+2az^8}{1.2.3.4.5.6.7.8.}$ &c; from which the ſine of A + z being ſubtracted leaves the Sine of A — z; and *vice verſa* the Sine of A — z being ſubtracted therefrom leaves the Sine of A + z; as will appear thus.

From $2a \frac{-2az^2}{1.2.} \frac{+2az^4}{1.2.3.4.} \frac{-az^6}{1.2.3.4.5.6.} \frac{+2az^8}{1.2.3.4.5.6.7.8.}$ &c, take S, A + z $= a \frac{+bz}{1.} \frac{-az^2}{1.2.} \frac{-bz^3}{1.2.3.} \frac{+az^4}{1.2.3.4.} \frac{+bz^5}{1.2.3.4.5.} \frac{-az^6}{1.2.3.4.5.6.}$ &c; and there will remain $a \frac{-bz}{1.} \frac{-az^2}{1.2.} \frac{+bz^3}{1.2.3.} \frac{+az^4}{1.2.3.4.} \frac{-bz^5}{1.2.3.4.5.} \frac{-az^6}{1.2.3.4.5.6.}$ &c. which before was ſhewn to be the Sine of A — z. 'Tis evident that this ſeries being Subtracted from the ſame remainder will leave the ſeries for the Sine of A + z. And hence the following rule is derived for finding the Sine of A + z, or A — z; the ſine of one of them and of the arc A being given.

Rule.

Three Arcs in Arithmetical proportion being given, from double the ſine of the middle arc ſubtract the ſecond difference, and from the remainder ſubtract the given Sine of one of the extremes, whether it be the greater or the leſſer, and the remainder will be the Sine of the other extreme.

Example.

Let it be required to find the Sine of 30°, 1′, the Sines of 30°, and 29°, 59′ being given.

Here 30° is the Mean arc; it's ſine is .50000000000; and the Sine of 29°, 59′, one of the extremes, is .49974806226; and the length of the arc z, to wit, 1 minute, is .000290888208; which ſquared and multiplied by the ſine of 30° the Mean arc, gives az^2, or $\frac{2az^2}{1.2.}$, the firſt term of the ſecond difference, = .000000042307, which being ſubtracted from 1, which is double the Sine of the mean arc, the remainder will be .999999957693; from which ſubtract the ſine .49974806226 of 29°, 59′, and there will remain .50025189943 for the ſine of 30°, 1′, the other extreme.

Mr. *Ham*, adds another example, and ſome remarks upon this way of making Sines; which as they ſeem not to be abſolutely neceſſary to the underſtanding the Method itſelf, are here omitted. Whoever is deſirous of ſeeing them, may find them in *Cunn*'s Euclid, Appendix. Pag. 366. much more upon this Subject may be ſeen in the Treatiſe of Trigonometry publiſhed in *Sherwin*'s Tables, where ſeveral different methods of making Sines, both by Serieſes and common Geometry, are delivered, and illuſtrated with examples.

The Reader will there find all the Powers of an arc of 1 Minute, or of z, ready calculated to ſeveral places in decimals. *Pag.* 47.

If the Arc A $=o$, then it's ſine $a=o$, and A $+z=z$. Wherefore the ſeries expreſſing the Sine of A $+z$ ought to be converted into the *Newtonian* ſeries for the Sine of z: and ſo we find it is; for if we take away the terms wherein a is a factor, and inſtead of b, which is now the coſine of o and therefore $=1$, ſubſtitute 1, the Series $a\frac{+bz}{1}\;\frac{-az^2}{1.2.}\;\frac{-bz^3}{1.2.3.}\;\frac{+az^4}{1.2.3.4.}\;\frac{+bz^5}{1.2.3.4.5.}\;\frac{-az^6}{1.2.3.4.5.6.}\;\frac{-bz^7}{1.2.3.4.5.6.7.}$ &c. will become $z\frac{-z^3}{1.2.3.}\;\frac{+z^5}{1.2.3.4.5.}\;\frac{-z^7}{1.2.3.4.5.6.7.}$ &c. And the Series for the Coſine of A $+z$, namely, $b\frac{-az}{1.}\;\frac{-bz^2}{1.2.}\;\frac{+az^3}{1.2.3.}\;\frac{+bz^4}{1.2.3.4.}\;\frac{-az^5}{1.2.3.4.5.}\;\frac{-bz^6}{1.2.3.4.5.6.}\;\frac{+az^7}{1.2.3.4.5.6.7.}\;\frac{+bz^8}{1.2.3.4.5.6.7.8.}$ &c. will be $1\frac{-z^2}{1.2.}\;\frac{+z^4}{1.2.3.4.}\;\frac{-z^6}{1.2.3.4.5.6.}\;\frac{+z^8}{1.2.3.4.5.6.7.8.}$ &c. $=$ CoS. z.

In like manner if A $=90°$, A $-z$ will be the complement of the arc z to a quadrant, and conſequently the Sine of A $-z$ ought to be the ſame with the Coſine of z: And ſo we find it is; for in this caſe, $a=1$, and $b=o$; wherefore the Series expreſſing the Sine of A $-z$, to wit, $a\frac{-bz}{1.}\;\frac{-az^2}{1.2.}\;\frac{+bz^3}{1.2.3.}\;\frac{+az^4}{1.2.3.4.}\;\frac{-bz^5}{1.2.3.4.5}\;\frac{-az^6}{1.2.3.4.5.6}$

$+b$

$\frac{+bz^7}{1.2.3.4.5.6.7.}\ \frac{+az^8}{1.2.3.4.5.6.7.8.}$ &c. becomes 1 $\frac{-z^2}{1.2.}\ \frac{+z^4}{1.2.3.4.}\ \frac{-z^6}{1.2.3.4.5.6.}\ \frac{+z^8}{1.2.3.4.5.6.7.8.}$ &c. = CoS. z. And the Coſine of A — z ought to be, and accordingly appears to be the ſame with the Sine of z. for $b\ \frac{+az}{1.}\ \frac{-bz^2}{1.2.}\ \frac{-az^3}{1.2.3.}\ \frac{+bz^4}{1.2.3.4.}\ \frac{+az^5}{1.2.3.4.5.}\ \frac{-bz^6}{1.2.3.4.5.6.}\ \frac{-az^7}{1.2.3.4.5.6.7.}\ \frac{+bz^8}{1.2.3.4.5.6.7.8.}$ &c. becomes in this caſe = z $\frac{-z^3}{1.2.3.}\ \frac{+z^5}{1.2.3.4.5.}\ \frac{-z^7}{1.2.3.4.5.6.7.}$ + &c √, = S, z.

As it may perhaps be a ſatisfaction to the curious reader, to ſee a geometrical proof, that the Second difference of the Sines of the three arcs A — z, A, A + z, or BE, BD, BF, (*fig.* 4,) is equal to the product of twice the Sine of the middle arc BD into the verſed ſine of the common difference DE of the three arcs, the radius being 1; or in all caſes, to that product divided by the radius; we have here ſubjoined one that is derived immediately from the contemplation of the 4th figure. Let EG and DS drawn parallel to CB meet DK and FI in T and S. 'Tis evident, DT or GS is the difference of the Sines EL, DK; and FS the difference of DK, FI. And becauſe GM = FM, GS muſt be = FM + MS, or FS + 2MS; whence GS — FS = 2 MS = $\frac{2DK \times DO}{CD}$, from the ſimilarity of the triangles CDK, ODV (V being the point where OM cuts DK), which gives this proportion, CD

$: DK :: DO : DV$; whence $\frac{DK \times DO}{CD} = DV = MS$.

And hence likewiſe we have a geometrical proof that if, according to the rule laid down above, we ſubtract the ſecond difference $2MS$ from $2DK$ or twice the ſine of the middle arc BD, and from the remainder ſubtract one of the extreme ſines FI or EL, this laſt remainder will be the other extreme ſine.

For $2DK - 2MS - FI = 2IS - 2MS - FI = 2IM - FI = 2IM - IM - FM = IM - FM = IM - MG = IG = EL$.

And $2DK - 2MS - EL = 2IM - IG = IM + MG = FI$.

PROP. 11. *Theorem.*

44. In a right-angled triangle if the Hypotenuſe is made the radius of a circle, the Sides will be the Sines of the oppoſite angles: but if one of the legs is made the Radius, the other leg will be the tangent of the angle to which it is oppoſite, and the Hypotenuſe will be the ſecant of the ſame angle. *Fig.* 10, and 11.

It is evident in *Fig.* 10, where AC the Hypotenuſe is the radius, that CB is the Sine of the arc CD, or the oppoſite angle at A; and that AB is the Coſine of that arc, or the Sine of the complement of CD to a quadrant, or of the angle A to a right angle, that is, of the angle ACB, to which it is oppoſite.

And in *Fig.* 11. If AB is made the Radius, BC will be the tangent of the arc BD, or the angle at A to which it is oppoſite, and the Hypotenuſe AC will be the Secant of that angle. If CB is made the Radius, AB will be the tangent of the arc BE, or the angle C to which it is oppoſite, and CA the Hypotenuſe will be the the Secant of the ſame. QED.

45. COROL. 1. Hence of the lengths of AC, and CB, or AC, AB, or AB, CB, are known in feet or inches or any other meaſure, the angles may be diſcovered by the following proportions. AC : AB :: R : S, C, whence the ſine of C is known, and conſequently the angle C may be found by ſeeking the Sine in a table of Sines and tangents.

Again, AC : CB :: R : S, A. And AB : BC :: R : T, A; and BC : AB :: R : T, C. Whence the tangents and Sines (and conſequently the angles) are diſcovered by the rule of Three.

46. COROL. 2. If the angles are known, and likewiſe the length of one of the ſides, or the Hypotenuſe, the other ſides may be found by the rule of Three.

For we may now invert the former proportions, and ſay, as S, C : R :: AB : AC, and S, A : R :: CB : AC; and T, A : R :: BC : AB; and laſtly, T, C : R :: AB : BC.

By the firſt of theſe analogies, from AB known we find the Hypotenuſe AC; by the ſecond we find it from BC ſuppoſed to be known. By the third, from BC known AB is diſcovered; and by the fourth from AB known BC is diſcovered.

Note, if one of the acute angles of a right-angled triangle is known, the other is known likewiſe, being it's complement to a right angle. (*El.* 1. 32. COR. 3.)

PROP.

PROP. 12. *Theorem.*

47. The Sides of Plane triangles are to one another in the ſame proportion as the Sines of the oppoſite angles. *Fig.* 12, 13, and 14.

Deſcribe about the plane triangle ABC the circle ABC by (*El.* 4. 5.) And from the center D draw the lines DE, DF, DG, perpendicular to the ſides of the triangle; which will therefore be biſected in E, F, G; (by *El*, 3. 3) Then will theſe halves of the ſides be the ſines of the oppoſite angles of the triangle, meaſured in the circumſcribing circle.

For in the acute-angled triangle, *Fig.* 12, the angle BDC at the center is double of the angle BAC at the circumference ſtanding upon the ſame arch BC, (by *El.* 3. 20). Therefore BDE, which (is equal to EDC (by *El.* 1. 8.) and therefore) is $= \frac{1}{2}$ BDC, is = to BAC. therefore BE, or $\frac{1}{2}$ BC, is the Sine of BAC. For the ſame reaſon BF is the Sine of the angle BCA, and AG of ABC.

Fig. 13. If the triangle is right-angled, the Hypotenuſe AC muſt be the diameter of the circle (*El.* 3. 31.), and half the Hypotenuſe BC, to wit, BD, will be the radius. But the radius is the Sine of a right angle; therefore BD, or $\frac{1}{2}$ BC, is the Sine of the oppoſite angle BAC. And BF (being the Sine of BDF, or of $\frac{1}{2}$ BDA) is the ſine of ACB; and for the ſame reaſon AG is the ſine of ABC.

In the obtuſe-angled triangle, *Fig.* 14, draw BL, CL, to any point L, in the arch BC; the angle L will (by *El.* 3. 22.) be the ſupplement of BAC to two right angles. Therefore it's ſine will be the ſame with that of BAC. Now becauſe the angle BDE ($= \frac{1}{2}$ BDC) is = L, it's ſine BE will be the ſine of L, and therefore of BAC. And BF (the ſine of BDF, or $\frac{1}{2}$ BDA) is the ſine of ACB; and for a like reaſon AG of ABC. Therefore in every triangle the halves of the Sides are the Sines of the oppoſite angles; conſequently the whole ſides are always proportional to the Sines of the oppoſite angles. QED.

PROP. 13.

48. If in a plane triangle any one of the ſides be called the Baſe, and the others the legs of the triangle; the ſum of the legs will be to the difference of the legs, as the tangent of the ſemiſum of the angles at the Baſe to the tangent of their ſemidifference.

Let the triangle be ABC (*Fig.* 15.) whoſe baſe is AC, and legs AB, BC. In AB produced take BH = BC, and BI = AB, (the ſide AB being ſuppoſed leſs than BC.) By *El.* 1. 32, the angle HBC is = A + ACB, the ſum of the angles at the baſe. Draw BE biſecting HBC, and EBC will be the ſemiſum of thoſe angles. And becauſe BC, BE are reſpectively equal to HB, BE, and the angle EBC = HBE, the triangles HBE, EBC will (by *El.* 1. 4.) be every way equal and alike, and the angle BEH = BEC. Therefore (*El.* 1. Def. 10.) BEC is a right angle. Conſequently if BE be made radius of a circle, EC will be the tangent of EBC in that circle; (*El.* 3. 16.)

Draw BD, and IG parallel to AC; and taking EF = ED, join B, F. Since BE, ED, are reſpectively equal to BE, EF, and BED = BEF, the triangles BEF, BED are every way equal and alike, and ang. EBF = EBD. But HBE = EBC; therefore HBF is = DBC, that is, *(El.* 1. 29.) = ACB. And HBD = BAC (by the ſame prop.) Therefore FBD is the difference of BAC and ACB the angles at the baſe; and EBD is half that difference, the tangent whereof is ED.

Since IG, BD, are parallel to AC, the triangles AHC, BHD, IHG, are ſimilar, and (by *El.* 6. 2.) AB : BH :: CD : DH, and BH : BI :: DH : DG, wherefore *ex æquo* AB : BI :: CD : DG. But by the conſtruction AB = BI; therefore CD = DG. But becauſe EC = EH, and ED = EF, CD muſt be = HF; therefore DG = HF; and (taking away the common part GF) HG = DF. Wherefore AH : IH :: (:: CH : GH :: ½ CH : ½ GH) :: EC : ED. QED.

Another Demonſtration of the ſame.

49. Making B your center *(Fig.* 16.) and BC, the longer of the two legs BC, BA, the radius, deſcribe a circle cutting BA produced in D and E. 'Tis evident DA is the ſum, and AE the difference of the legs. Join D, C, and E, C. The ſum of the angles BAC, BCA, at the baſe AC is equal to the ſum of the angles BEC, BCE, at the baſe EC of the iſoſceles triangle BEC, and conſequently *(El.* 1. 5) to twice BEC. Whence BEC, or it's equal BAH (drawing AH parallel to EC) is the ſemiſum of the angles at the baſe.

The angle BAC is (= BAH + HAC = BEC + ACE = BCE + ACE) = BCA + 2ACE. Whence BAC — BCA = 2ACE, and the ſemi-difference of BAC, BCA, is = ACE or HAC.

But DHA is a right angle, being equal to DCE in the ſemicircle DCE. Therefore DH, HC are the tangents of BAH, or DAH, and HAC, the radius being AH. And from the ſimilarity of the triangles DHA, DCE, we have DA : AE : : DH : HC. QED.

PROP. 14.

50. If from the vertex of one of the angles of a plain triangle a line be drawn perpendicular to the oppoſite ſide, or Baſe, and dividing it into two ſegments; the Baſe will be to the ſum of the ſides, as the difference of the ſides to the difference of the ſegments of the Baſe.

In an obtuſe angled triangle as BCD, *Fig.* 17, 'tis evident the perpendicular muſt be drawn from the vertex of the obtuſe angle; otherwiſe it will not fall within the triangle, and divide the baſe into two ſegments. In an acute-angled triangle as BCD, *Fig.* 18, it may be drawn from the vertex of any one of the angles. See *El.* 2, *Prop.* 12, & 13.

Let BCD be a plane triangle, DC it's baſe, and BE a line drawn perpendicular to DC from the point B, and dividing it into two ſegments, CE, ED. With B as a center, and BC the leſſer ſide as a Radius, deſcribe a circle cutting the baſe in F, and the other ſide BD in H. Produce DB till it meets the circumference in G; and becauſe BG and BH are each of them equal to

BC, DG will be the ſum of the legs, and DH their difference. And becauſe CE is = EF (*El.* 3. 3.) DF will be the difference of the ſegments DE, EC of the Baſe.

I ſay, DC : DG :: DH : DF.

For (by *El.* 3. 36. COROL.) the rectangle under DC, DF is equal to the rectangle under DG, DH; wherefore (*El.* 6. 16.) DC : DG :: DH : DF. QED.

Another Demonſtration of the ſame.

From the angular point B in the triangle BCD (*Fig.* 19, and 20.) draw BE perpendicular to the baſe CD. And with B as a center and BD the longer of the ſides as a radius, deſcribe a circle cutting BC and DC produced in M, K, L. Since BK, and BM are each of them equal to BD, CM will be the ſum, and CK the difference of the ſides BD, BC. And becauſe DE = LE (*El.* 3. 3.) LC is the difference of the ſegments DE, EC, of the Baſe.

Now the triangles MCD, LCK, are ſimilar; becauſe the angles MCD, LCK are vertical; and CMD, CLK are equal, as ſtanding on the ſame arch KD. Whence we have this proportion, DC : CM :: CK : LC. QED.

PROP. 15. *A Problem.*

The Sum and the difference of two quantities being given, to find the quanties themſelves.

If to the ſemiſum be added the ſemidifference, the aggregate will be the greater quantity; and if from the ſemiſum be taken the ſemidifference, the remainder will be the leſſer.

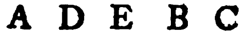


Fig. 21.

For let AB, BC (*Fig.* 21.) repreſent the two quantities. Take AD = BC. And DB will be their difference. Biſect their Sum AC in E; and AE, or EC will be the ſemiſum, and DE, or EB the ſemidifference.

Now AB the greater quantity is = AE + EB; and BC the leſſer is = EC — EB. QED.

This will appear alſo, if we expreſs the quantities Algebraically, and call the greater a, and the leſſer b. For $\frac{a+b}{2.} + \frac{a-b}{2.} = \frac{2a}{2} = a$. And $\frac{a+b}{2.} \; \frac{-a+b}{2} = \frac{2b}{2} = b$.

53. We come now to apply the laſt five propoſitions to the ſolution of the caſes of plane triangles. But firſt it will be proper to obſerve, that the data of ſome of the caſes coincide; to wit, 'tis the ſame thing to have two angles of a triangle given, as to have all the three; becauſe when two of them are given, the third is given likewiſe, being the complement of the ſum of the former to two right angles.

But in a right-angled triangle if only one of the acute angles is given, the other angles are given likewiſe, one of them being a right angle, and the other the complement of the given angle to a right one; as was obſerved in *Art.* 46.

The caſe wherein two ſides of a right-angled triangle are given, and 'tis required to find the third, may be ſolved immediately by the firſt propoſition, and therefore is not ſet down among theſe that follow.

The Trigonometrical *Solutions of a Right-angled Triangle. (Fig.* 22.)

	Given	Sought	
1	AB, BC the legs.	the angles	AB : BC :: R : T, A; the complement of which to a right angle is the angle C.
2	AB, AC, a leg and the Hypotenuſe	the angles	AC : AB :: R, : S, C; the complement of which to a right angle is the angle A.
3	AB, and A, a leg and an angle.	BC the other leg.	R : T, A :: AB : BC
4	AB, and C, a leg and an angle.	AC the Hypotenuſe.	S, C : R :: AB : AC

The Trigonometrical *Solutions of Oblique-angled Triangles.*

	Given	Sought	
1	A, B, C, and AB, the angles and a ſide. (*Fig.* 23, 24.)	BC, AC, the other ſides.	S, C : S, A :: AB : BC; and S, C : S, B :: AB : AC. By what was obſerved in *Art.* 53, the caſe wherein two angles and a ſide are given falls under this.

Given

	Given	Sought	
2	A, B, C, all the angles; or only A, B, two of the the angles. (*Fig.* 23,24.)	the proportions of AB, AC, BC, the ſides.	S, C : S, A :: AB : BC S, C : S, B :: AB : AC. 'Tis evident that if only the angles are given, only the proportion of the ſides, and not the ſides themſelves, can be diſcovered.
3	AB, BC, and C; two ſides, and an angle oppoſite to one of them.	A, and B, the remaing angles.	AB : BC :: S, C : S, A, which therefore may be found. But, becauſe every angle has the ſame ſine as it's complement to two right angles, we muſt firſt know the ſpecies of A, i. e. whe-it be acute or obtuſe. In *Fig.* 25. BA, BA2, are made equal, and conſequently the angles A and BA2A are equal. But BA2A, has the ſame ſine as BA2C, being it's complement to two right angles. Therefore A, has the ſame ſine as BA2C; that is, the ſine diſcovered by the proportion may belong either to the acute angle A, or the obtuſe angle BA2C.

	Given	Sought	
			N. B. If C, the known angle is either a right or an obtuſe angle, the angle oppoſite to BC muſt be acute. If C is a right angle, this caſe falls under caſe 2d of right angled triangles.
4	AB, BC, and B, two ſides and the angle between them. Or, the proportion only of AB, BC, two of the ſides, and the angle B between them. *Fig.* 23, 24.	Angles A and C.	Since the angle B is given, the ſum of the angles A and C at the Baſe AC is given likewiſe, being the complement of B to two right angles. Therefore their ſemiſum and it's tangent is given: Say therefore (by P. 13.) $BC+AB : BC-AB :: T, \frac{A+C}{2} : T, \frac{A-C}{2}$: Whence the tangent of the ſemidifference is known. Therefore the ſemidifference may be found by a table of ſines and tangents. Conſequently the angles themſelves may be found by *P.* 15.

	Given	Sought	
5	AB, BC, AC, all the ſides. *Fig.* 26, 27.	the angles.	Drop a perpendicular from the vertex A to the Baſe BC; and ſay, (by *P.* 14.) as BC : AC+AB :: AC—AB : DC—DB, the difference of the ſegments. Their ſum is known, being equal to the baſe itſelf. Wherefore (by *P.* 15.) the ſegments themſelves may be found. And conſequently, by reſolving the right-angled triangles ABD, ACD, in each of which a leg and the Hypotenuſe is given (to wit, in ABD, BD and AB, and in ACD, DC and AC) by caſe 2d of right-angled triangles, the angles may be diſcovered.

As an example to one of theſe caſes, let the proportion of BC to BA be that of 467 to 1; (*Fig.* 24.) and the angle CBA = 51°, 46′. 'Tis required to find the angles BAC, ACB.

Their ſum is (= 180° --- 51°, 46′) = 128°, 14′, and ſemiſum = 64°, 7′, the tangent whereof is 2.0609442. Say therefore as 468 : 466 :: 2.0609442 : 2.0521367 = tangent 64°, 2′.

Subtract therefore from the ſemiſum 64°, 7′, the ſemidifference 64°, 2′, and the remainder will be the leſſer angle ACB = 5′. And add 64°, 2′ to 64°, 7′, and the ſum will be the greater angle BAC = 128°, 9′.

The angle ACB, or 5′, is the angle contained between the real direction in which a body placed on the ſurface of the earth in latitude 51°, 46′ tends towards the earth (taking in the conſideration of the centrifugal force) and a radius of the earth paſſing through the body. *Keill* againſt *Burnet*, *Page* 101. (*Fig.* 7.)

g. 11.

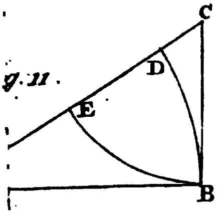


Fig. 12.

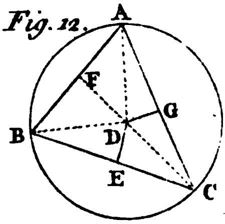
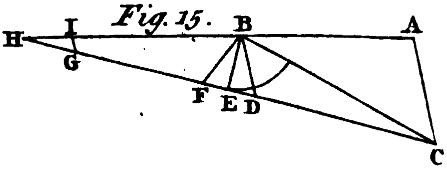


Fig. 15.



ig. 17.

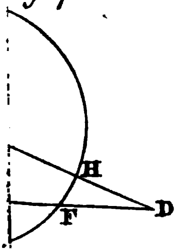


Fig. 18.

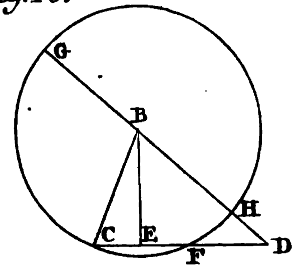


Fig. 20.

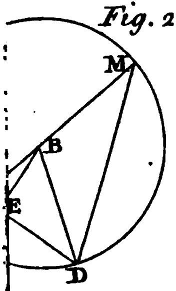
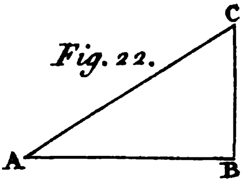


Fig. 22.



ig. 24.

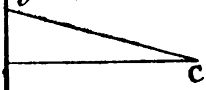
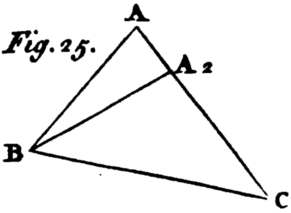


Fig. 25.



PART the SECOND.

Containing a few Propoſitions concerning the moſt remarkable properties, and relations of the Lines in and about a circle to one another.

54. PROP. 16.

THE Secants of any two angles are to one another reciprocally as the coſines of thoſe angles. *Fig.* 28.

For (by *Art.* 13.) the radius AB is a mean proportional between the ſecant AC and the coſine AF of the arc BK, and likewiſe between the ſecant AD, and the coſine AH of the arch BG. Whence the rectangle AH x AD, is equal to the rectangle AC x AF; each rectangle being equal to the ſquare of the Radius. Therefore AD : AC :: AF : AH. QED.

Or, it may be proved thus. Produce FK 'till it meet AD in E. And ſince the triangles CAD, KAE, LAG, are ſimilar; and likewiſe FAK, HAL, we have theſe proportions. AD : AC :: AE : AK :: AG : AL :: AK : AL :: AF : AH. QED.

Or thus. (By *Art.* 47.) AD : AC :: S, ACD : S, ADC :: S, ACB : S, ADC :: S, KAN : S, GAN :: AF : AH. QED.

PROP. 17.

55. The tangents of any two angles are to one another in a ratio compounded of the direct ratio of the ſines, and the inverſe of the coſines.

The tangent BD (*Fig.* 28.) is to the tangent BC, in a ratio compounded of the ratios of BD to BA, and BA to BC; that is, of GH to HA, and AF to FK; or as GH x AF to HA x FK; or in a ratio compounded of the ratios of GH to FK, and AF to HA. QED.

Or it may be proved thus. Let the angles be CBT, CBV; (*Fig.* 29.) their tangents CT, CV. I ſay, CT is to CV, in a ratio compounded of CD to CE, directly, and BD to BE inverſely.

For the triangles BCT, BDC are equiangular, the angles at D, and C being right ones, and that at B common to them both. Therefore CT : CB :: CD : BD. Again, becauſe the right angled triangles BCV, BCE, are equiangular, we have CB : CV :: EB : EC. Therefore the ratio of CT to CV (which is compounded of the ratios of CT to CB, and of CB to CV) is equal to the ratio of the rectangle under CD, EB, to the rectangle under BD, EC, or (*El.* 6. 23.) is compounded of the ratios of CD to CE, and BE to BD, that is of the ſines directly and coſines inverſely. QED. This demonſtration is Dr. *Smith*'s in his optics.

56. COROL. 1. Hence the tangents of any two angles are to one another in a compound ratio of the ſines and ſecants.

57. COROL. 2. Hence the ſines of any two angles are directly as the tangents, and inverſely as the ſecants.

For inſtance, let there be two angles whoſe tangents are to one another as 2 to 1; as is the caſe with the angles of incidence and refraction in the primary rainbow: and let the tangent of the greater be T, and conſequently that of the leſſer $\frac{T}{2}$. The radius being 1, the ſecants will be $\sqrt{TT+1}$, and $\left(\sqrt{\frac{TT}{4}+1}=\right)\sqrt{\frac{TT+4}{4}}$ or, $\frac{\sqrt{TT+4}}{2}$. Hence the ratio of the ſines is that of $\frac{T}{\sqrt{TT+1}}$ to $\frac{T}{2}\times\frac{2}{\sqrt{TT+4}}$ or $\frac{T}{\sqrt{TT+4}}$; or of $\sqrt{TT+4}$ to $\sqrt{TT+1}$.

58. COROL. 3. The Secants are directly as the tangents and reciprocally as the Sines.

PROP. 18.

59. The chord AB of any arc AB (*Fig.* 30.) is a mean proportional between the verſed ſine of that arc, and the Diameter; the Sine BD of any arc AB is a mean proportional between the verſed ſine of that arc and the verſed ſine of it's complement to a ſemicircle; and the tangent BP

of any arc is a mean proportional between the ſum CP and difference AP of the radius and ſecant.

Draw the chord BC, and the radius BO. The angle CBA ſituated in the ſemicircle is a right angle. Therefore (*El.* 6. 8.) the triangles CBA, CBD, DBA, are ſimilar. Whence we have CA : AB :: AB : DA; which is the firſt part of the propoſition: and CD : BD :: BD : DA; which is the ſecond part.

Draw the tangent AE meeting BP in E. The angle EAB will = EBA, becauſe ABO, BAO, are equal, as being at the baſe of the iſoſceles triangle OBA; and OAE, OBE, are right angles. But EAB is = DBA (*El.* 1. 29.) and therefore to BCP. Therefore PBA = BCP; and the angle at P is common to both the triangles BCP, BAP; which are conſequently equiangular and ſimilar. Therefore CP : BP :: BP : AP. QED.

Note. The laſt part of the propoſition is proved by *Euclid*, in *El.* 3. 36. as indeed is the ſecond part in *El.* 6. 13. though not in the form of a theorem.

But as theſe properties of the ſine, chord, and tangent, bear a great reſemblance to each other, it was thought not improper to mention them here all together, and for variety's ſake to give another proof of the latter.

60. COROL. 1. The chord of any arc biſects the angle contained between the Sine and the tangent. For it appeared in the demonſtration, that EAB = EBA; and DBA = EAB. Whence DBA = EBA.

61. COROL. 2. The verſed ſines of any two arcs are proportional to the ſquares of the chords of thoſe arcs.

For they are equal to the ſquares of the chords applied to the diameter, which (by *El.* 6. 1.) are proportional to the ſquares of the chords.

62. COROL. 3. The verſed ſines of any two arcs are proportional to the ſquares of the ſines of half thoſe arcs; by *Art.* 5. 15. and *El.* 5.

Galileo, in his excellent mechanical dialogues, makes uſe of this propoſition to ſhew that no ſtring, or chain, how ſlender ſoever, can be perfectly ſtretched into a right line by any force however great. *See* Dial. 4. towards the end.

PROP. 19.

63. The rectangles under the tangent and cotangent, the coſine and ſecant, and the ſine and coſecant of any angle, are all equal to each other.

For each of them is equal to the ſquare of the radius, by *P.* 2. *Coroll.* 2. and 3. QED.

64. COROL. Hence the tangents of two angles, are to each other reciprocally as the cotangents. For the rectangle under the tangent and cotangent of each of the angles being equal to the ſquare of the radius, they muſt be equal to one another. Whence the tangent of the firſt angle is to the tangent of the ſecond, as the cotangent of the ſecond to the cotangent of the firſt.

PROP.

PROP. 20.

65. The difference ED (*Fig.* 31.) between the diameter AD of a circle, and the chord AB of any arc in it; the chord FD of half the ſupplement of that arc to a ſemicircle; and the Radius FC; are continual proportionals.

For drawing AF, and EF, the triangles ABF, AEF, will be every way equal and alike; becauſe AF, AB, are reſpectively equal to AF, AE, and the angle BAF to the angle EAF, the arches BF, FD, being equal. Therefore EF is = FB, and conſequently to FD. Therefore (*El.* 1. 5.) FED is an iſoſceles triangle. And becauſe CDF is common to both the iſoſceles triangles CDF, FED, it follows that they are equiangular, and therefore ſimilar (*El.* 6. 4.) Whence ED : DF :: DF : FC. QED.

PROP. 21.

66. If one of the angles, as ABC (*Fig.* 32.) of a triangle ABC be biſected by a right line BE meeting the baſe in E; the ſquare of BE will be equal to the difference of the rectangles under the ſides AB, BC, and the ſegments AE, EC, of the baſe.

Deſcribe a circle round the given triangle; and producing BE till it cuts the circumference in D, draw the chord DC.

The triangles ABE, BDC, are ſimilar; becauſe the angles BAC, BDC ſtand upon the ſame arch BC, and conſequently are equal; and the angles ABD, DBC, are equal by the hypotheſis. Therefore

fore AB : BE : : DB : BC, whence AB x BC = DB x BE = (*El.* 2. 3.) DE x EB x BE*q*. But (by *El.* 3. 35.) DE x EB = AE x EC. Therefore AB x BC = AE x EC x BE*q*, and BE*q* = AB x BC — AE x EC. QED.

This is one of thoſe propoſitions Sir *Iſaac Newton* mentions as of exceeding great uſe in the application of Algebra to Geometry. *See* Arith. Univerſ. *P.* 104. Edit. 2d.

PROP. 22.

67. The ſine of the ſum of any two angles is to the ſine of their difference as the ſum of their tangents to the difference of their tangents.

Let the angles be ACF, ACB, (*Fig.* 33.) and their tangents AF, AB. Take AD = AB, BF, and DF will be the ſum and difference of the tangents; and DCF the difference of the angles. Draw DE parallel to BC, meeting CF in E. Then becauſe BCF, DEF are ſimilar triangles, we ſhall have BF : DF : : BC : DE, that is, as DC : DE; becauſe (by *El.* 1. 4.) BC, DC are equal. But DC : DE : : S, DEC : S, DCE : : S, DEF or BCF : S, DCF. Therefore BF : DF : : S, BCF : S, DCF. QED.

PROP. 23.

68. The ratios; of the ſum of the ſecant and tangent of a circular arch to the radius; of the ſum of the radius and ſine to the coſine; of the radius to the difference of the ſecant and tangent; of the coſine to the difference of the radius and

ſine; and of the radius to the tangent of half the complement of the given arch to a quadrant; are all equal to each other.

Let L rR (*Fig.* 34.) be a ſemicircle, whoſe diameter is LR, and center C; in the quadrant Rr take any arch RA, and draw AD, RP, and CP it's ſine, tangent, and ſecant; from r draw rG the tangent of the arch rA; and biſecting rA in E, let rF and CF be the tangent and ſecant of rE. We are to prove the ratios; of CP + RP to CR; of CA + AD to CD; of CR to CP — RP; of CD to CA — AD; and of Cr to rF; are all equal to each other.

With P as a center, and PR as a radius, draw the circular arch RM cutting CP in M; and with A as a center, and AD as a radius, draw the circular arch Dm cutting CP in m; and (by *El.* 3. 36.) we ſhall have $CRq = CM \times \overline{CP \times RP} = \overline{CP - RP} \times CP \times RP$, and $CDq = Cm \times \overline{CA \times AD} = \overline{CA - AD} \times \overline{CA \times AD}$. Therefore CP + RP : CR :: CR : CP — RP, and CA + AD : CD :: CD : CA — AD. But becauſe the triangles CPR, CAD are ſimilar, we have CP : RP :: CA : AD, and *componendo* CP + RP : RP :: CA + AD : AD; and we have likewiſe RP : CR :: AD : CD; therefore *ex æquo*, CP + RP : CR :: CA + AD : CD, and conſequently the ratios of CP + RP to CR, of CA + AD to CD, of CR to CP — RP, and of CD to CA — AD, are all equal to each other. Laſtly, from the ſimilarity of the triangles CPR, CGr, we have CP + RP : CR :: CG + Cr : Gr; but (by *El.* 6. 3.) CG is to Cr as GF to Fr; therefore *componendo*, CG + Cr : Cr :: Gr : Fr; and *permutando*, CG + Cr :

Cr : Gr :: Cr : Fr; conſequently Cr : Fr :: CP + RP : CR. QED.

COROL. Since by the foregoing proportions we have CR : CP — RP (:: CP + RP : CR) :: Cr or CR : rF, it follows that CP — RP is = rF, or that the difference of the ſecant and tangent of a circular arch is equal to the tangent of half the complement thereof to a quadrant.

PROP. 24.

69. The chord of the difference of any two arches is to the difference of their ſines, as the radius to the coſine of an arithmetical mean between the two arches.

Let the arches be BE, BF (*Fig.* 35.) and the arithmetical mean between them BD. FE is the chord of their difference; and FG the difference of their ſines EL, FI.

From the ſimilarity of the triangles FGE, CKD, it follows that, FE : FG :: CD : CK. QED.

70. COROL. 1. Hence if there be four arches BS, BV, BE, BF, and the difference SV of the two leſſer arches be equal to the difference FE of the two greater arches; the difference PV of the ſines of the leſſer will be greater than FG the difference of the ſines of the greater arches.

For CN being greater than CK, the radius CT, or CD is to CN in a leſs proportion than to CK; therefore the chord VS is to VP in a leſs ratio than the chord FE is to FG. And theſe chords are equal, becauſe the arches they ſubtend are ſo; therefore VP is greater than FG.

71. Corol. 2. Hence the ratio of FI to EL is leſs than that of VM to SO. For take a part Iy in the line FI (produced if need be) that ſhall be to EL as VM to SO; I ſay, Iy will be greater than IF, or the point y will fall above F. For ſince Iy : EL :: VM : S, it follows *dividendo*, that yG : EL :: VP : SO, and alternately, yG : VP :: EL : SO. But EL is greater than SO; therefore yG is greater than VP, and conſequently than FG; therefore yI is greater than FI. Therefore the ratio of FI to EL is leſs than that of yI to EL, or of VM to SO. QED.

72. Corol. 3. If there be four arches AF, AE, AV, AS, and the difference FE of the two leſſer be equal to the difference VS of the two greater, the ratio of the ſecants of AV, AS, the two greater arches will be greater than the ratio of the ſecants of AF, AE, the two leſſer arches. For Sec, AS : Sec, AV :: VM : SO; and Sec, AE : Sec, AF :: FI : EL. (by *P.* 16.)

Note. This is not always true of the tangents: with what limitations it may be affirmed of them likewiſe, will be ſhewn in *P.* 30. *Corol.* 9.

73. Corol. 4. The difference of the ſecants of the arcs AV, AS, bears a greater proportion to thoſe ſecants, than the difference of the ſecants of AF, AE, does to the ſecants of AF, AE, themſelves. For ſince VM, SO, are leſs than FI, EL; VP would bear a greater proportion to them than FG does to FI, EL, if VP, FG were equal. Therefore a *fortiori*, ſince VP is greater than FG, it muſt be in a greater proportion to VM or SO, than FG is to FI or EL.

74. COROL. 5. If there be four arches BS, BV, BE, BZ, of which the two latter are greater than the two former; and the ratio of the ſines VM, OS of the two leſſer be equal to the ratio of the ſines ZX, EL, of the two greater: the difference EZ of the two greater arches will be greater than VS, the difference of the two leſſer.

For ſince ZX : EL : : VM : SO, ZX muſt be to EL, in a greater proportion than that of FI to EL (by *Corol.* 2.) Therefore ZX is greater than FI. Therefore the arch ZE is greater than the arch FE; that is, than the arch VS.

Hence it is that the difference of the angles of incidence and refraction continually increaſes at the ſame time as thoſe angles themſelves increaſe. *Smith's* Optics. *Art.* 16.

75. COROL. 6. The ſecant of BZ, is to the ſecant of BE, in a greater proportion than that of the ſecant of BV, to the ſecant of BS.

For what was proved in *Corol.* 3. concerning the ſecants of AF, AE, AV, AS, is true of the ſecants of BS, BV, BE, BF, which are related to each other in exactly the ſame manner as the former. Therefore Sec, BF, is to Sec, BE in a greater proportion than Sec, BV to Sec, BS. But becauſe EZ is greater than FE (by *Corol.* 5.) BZ muſt be greater than BF; therefore the ſecant of BZ is greater than the ſecant of BF, and of conſequence bears a much greater proportion to the ſecant of BE, than the ſecant of BV to the ſecant of BS.

76. Corol. 7. The difference of the ſecants of BZ, BE is to each of thoſe ſecants themſelves in a greater proportion than the difference of the ſecants of BV, BS is to each of the ſecants of BV, BS, reſpectively.

For let the ſecants be $C\phi$, $c\pi$, $C\theta$, $c\delta$, (*Fig.* 36.) and ſince $c\phi$ is to $c\pi$ in a greater proportion than $c\theta$ to $c\delta$, take in $c\pi$ produced a line $c\alpha$ of ſuch a length, that $c\phi$ ſhall be to it as $c\theta$ to $c\delta$. Then *dividendo*, $c\phi - c\alpha : c\phi :: c\theta - c\delta : c\theta$; but $c\phi - c\pi$ is greater than $c\phi - c\alpha$: therefore $c\phi - c\pi : c\phi$ in a greater proportion than $c\theta - c\delta$ to $c\theta$.

Again, Since $c\phi : c\alpha :: c\theta : c\delta$, it follows *dividendo*, that $c\phi - c\alpha : c\alpha :: c\theta - c\delta : c\delta$. But $c\phi - c\pi$ is greater than $c\phi - c\alpha$; and $c\alpha$ is greater than $c\pi$: therefore on a double account $c\phi - c\pi : C\pi$ in a greater proportion than $c\theta - c\delta$ to $c\delta$.

77. Corol. 8. The tangent $B\phi$ is to the tangent $B\pi$ in a greater proportion than that of the tangent $B\theta$ to the tangent $B\delta$.

For (by *P.* 17. *Corol.* 1.) $B\phi : B\pi :: c\phi \times zx : C\pi \times EL$; but by the hypotheſis, $zx : EL :: VM : SO$. Therefore $B\phi : B\pi :: c\phi \times VM : c\pi \times SO$. But $B\theta : B\delta :: c\theta \times VM : c\delta \times SO$. And (by *Corol.* 6.) the ratio of $c\phi$ to $c\pi$ is greater than that of $c\theta$ to $c\delta$. Conſequently the ratio of $c\phi \times VM$ to $c\pi \times SO$, (which is equal to the ſum of the ratios of $c\phi$ to $c\pi$ and VM to SO) muſt be greater than the ratio of $c\theta \times VM$ to $c\delta \times SO$. (which is equal to the ſum of the ratios of $c\theta$ to

$c\delta$ and VM to SO); that is, the ratio of $B\phi$, $B\pi$ is greater than that of $B\theta$, $B\delta$.

78. Corol. 9. The difference $\pi\phi$ of the tangents $B\phi$, $B\pi$, is to each of thoſe tangents in a greater proportion than that of the difference $\delta\theta$ of the tangents $B\theta$, $B\delta$, to each of the tangents $B\theta$, $B\delta$, reſpectively.

For ſince $B\phi : B\pi$ in a greater ratio than that of $B\theta$ to $B\delta$, take $B\gamma$ of ſuch a length that $B\phi : B\gamma :: B\theta : B\delta$. Then *dividendo*, $\phi\gamma : B\phi :: \theta\delta : B\theta$. But $\pi\phi$ is greater than $\phi\gamma$. Therefore $\pi\phi : B\phi$ in a greater ratio than $\theta\delta$ to $B\theta$.

Again, ſince $B\phi : B\gamma :: B\theta : B\delta$. It follows *dividendo* that $\phi\gamma : B\gamma :: \theta\delta : B\delta$. Therefore $\pi\phi : B\gamma$ in a greater ratio than $\theta\delta$ to $B\delta$. But $B\pi$ is leſs than $B\gamma$; therefore a *fortiori* $\pi\phi : B\pi$ in a greater ratio than $\theta\delta$ to $B\delta$.

79. I hope I ſhall be excuſed if I take the liberty of explaining by the help of this corollary a paſſage in Dr. *Smith*'s Optics, which to me appeared to be ſomewhat difficult, and I imagine will be thought ſo by ſuch others of the readers of that learned author, as are but moderately ſkilled in theſe ſciences. In his ſecond book, Ch. 9. *P.* 4. *Corol.* 4. *Art.* 420, he has theſe words; "hence it appears that while the angle of incidence continually increaſes, the focal diſtance Bf continually decreaſes, &c." It is proved in the propoſition that as the difference of the tangents of incidence and refraction is to the tangent of incidence, ſo is the coſine of refraction BE to the focal diſtance Bf. Now when the angle of incidence increaſes, the proportion of the difference of the tangents to the tangent of incidence

increaſes likewiſe; therefore if BE were to remain the ſame, Bf muſt decreaſe. But BE decreaſes; therefore a *fortiori* BF muſt decreaſe. And becauſe, when a ray of light paſſes out of a denſer medium into a rarer, the angle of refraction is greater than the angle of incidence, 'tis evident when the angle of refraction is a right angle, or the greateſt poſſible, the angle of incidence will be leſs than a right angle; therefore it's tangent will be finite. But the tangent of refraction will be infinite; conſequently it's exceſs above the tangent of incidence will be infinite likewiſe. Therefore the ratio of the difference of the tangents to the tangent of incidence is that of a quantity infinitely great to a finite one. Therefore if BE were of a finite magnitude, BF would be equal to nothing; but BE (being now the coſine of a right angle is itſelf equal to nothing; therefore a *fortiori*, BF muſt be nothing likewiſe.

When a ray of light paſſes out of a rarer medium into a denſer, the angle of incidence is greater than that of refraction; conſequently when the former becomes a right angle, or the greateſt poſſible, and it's tangent infinite, the latter will be leſs than a right angle, and have a finite tangent. The difference of the tangents therefore will be infinite, and it's laſt ratio to the tangent of incidence will be a ratio of equality. Therefore in this caſe the focal diſtance BF muſt be equal to the coſine of refraction BE.

80. *Note.* If there be three quantities, as the numbers 2, 3, 4, of which the firſt is leſs than the ſecond, and the ſecond than the third, the ſecond is uſually ſaid to be to the third in a greater ratio

than the firſt to the third; thus 3 is ſaid to be to 4 in a *greater* proportion than 2 to 4; and ſuch kind of expreſſions have been uſed in the preceeding corrollaries. But the more accurate expreſſion would be, that the ratio of 3 to 4 is a leſs ratio *minoris inæqualitatis*, or of minority, than the ratio of 2 to 4. For in ſtrictneſs the ratio of 2 to 4 is greater than that of 3 to 4, and accordingly has a greater logarithm. It is much the ſame kind of impropriety to ſay the ratio of 3 to 4 is *greater* than that of 2 to 4, as to ſay that —12 is *greater* than —16, (as in *Saunderſon*'s Algebra, *Art.* 113.) whereas it is but three quarters of it. In both caſes the word *greater* is uſed in different ſenſes; for when we ſay the ratio of 2 to 4 is *greater* than that of 3 to 4, we then underſtand by *greater* more diſtant from a ratio of equality; and when we ſay, the ratio of 3 to 4 is *greater* than that of 2 to 4, we then eſtimate the greatneſs of the ratio by it's nearneſs to a ratio of majority. In the ſame manner when —16 is ſaid to be *greater* than —12, as for inſtance a velocity that ſhall carry a body uniformly at the rate of 16 feet in a ſecond from *Weſt* to *Eaſt*, than a velocity that ſhall carry it through only 12 feet in the ſame time from *Weſt* to *Eaſt*, regard is had only to the nature and proportion of theſe velocities, without any relation to a velocity the contrary way.

But when —12 is ſaid to be *greater* than —16, or the velocity of 12 feet *Eaſt-ward*, than the velocity of 16 feet *Eaſt-ward*; all that is meant is only, that the velocity of 12 feet *Eaſtward* approaches nearer to, or differs leſs from, an affirmative velocity, or velocity *Weſt-ward*, then the velocity of 16 feet *Eaſt-ward*. In ratios of majority, and in affirmative quantities, theſe different ſenſes of the word *greater* always go together; thus 4 is to 2 in a *greater* proportion than to 3 in both ſenſes; and likewiſe 16 is greater then 12 in both ſignifications. It would certainly be better to uſe the former of theſe expreſſions, as Dr. *Smith* has done in *Art.* 602. of his Optics, where he calls the ratio of an infinitely ſmall quantity to a finite one, not an infinitely ſmall ratio, but an infinitely great ratio of minority. But as this degree of accuracy is not to be found in moſt writers, but the other expreſſion is far the more common of the two, it is at leaſt very allowable to make uſe of it; particularly as inaccuracies of this kind, when one is aware of them, cannot poſſibly occaſion any confuſion or miſtakes.

Concerning the nature of ratios ſee *Saunderſon*'s Diſcourſe on their Compoſition and Reſolution, in his Algebra, Book 7, from *Art.* 292, to *Art.* 305.

PROP. 25.

The tangent of the ſum of two angles is to the ſum of their tangents, as the ſquare of the radius to the ſquare of the radius diminiſhed by the rectangle under the tangents: And the tangent of the difference of any two angles is to the difference of their tangents, as the ſquare of the radius to the ſquare of the radius increaſed by the rectangle under the tangents.

PART 1.

Let RA, RB (*Fig.* 37.) be the tangents of any two angles ROA, ROB; I ſay, the tangent of the ſum of ROA, ROB is to AB, the ſum of the tangents, as ROq to $ROq - \overline{RB \times RA}$.

For as AB is to AO, the ſecant of either of the angles, ſo let AO be to AC, to be placed from A towards B. Again, as RC is to RO, ſo let RO be to RD; and RD will be the tangent of the ſum. For, joining CO; by the firſt of theſe proportions the triangles AOB, AOC, have the ſides about their common angle at A proportional, and conſequently are ſimilar, *El.* 6. 6. therefore AOB = ACO. And by the ſecond proportion, the triangles ROC, ROD have the ſides about the right angles CRO, DRO, proportional, and conſequently are ſimilar; therefore RCO = ROD. But RCO or ACO = AOB, by what was juſt now proved; therefore ROD is = AOB, the ſum of the angles ROA, ROB; and RD is the tangent of that ſum.

And ſince by the firſt proportion, $AC = \frac{AOq}{AB}$ $= $ (by *El.* 1. 47.) $\frac{RAq+ROq}{RB+RA}$, it follows that $RC\ (= AC - RA) = \frac{RAq+ROq}{RB+RA} - RA = \frac{ROq - RB \times RA}{RB+RA}$. Whence RD (which by the ſecond proportion is $= \frac{ROq}{RC} = \frac{ROq \times \overline{RB+RA}}{ROq - \overline{RB \times RA}}$ and (multiplying by $ROq - \overline{RB \times RA}$) we have $RD \times \overline{ROq - RB \times RA} = ROq \times \overline{RB+RA}$.) Whence $RD : \overline{RB+RA}$, or $AB :: ROq : ROq - \overline{RB \times RA}$. QED.

PART 2.

Let RA, RB (*Fig.* 38.) be tangets, as before, of any two angles ROA, ROB. I ſay, the tangent of the difference of ROA, ROB, is to AB the difference of their tangents, as RO*q*, to $ROq + RB \times RA$. For here, again, as AB to AO, ſo let AO be to AC, to be placed from A towards B. Again, as RC to RO, ſo let RO be to RD; and RD will be the tangent of the difference of ROA, ROB. For by the firſt proportion, (*El.* 6. 6.) the triangles AOB, ACO are equiangular; therefore AOB = ACO. And by the ſecond proportion, the triangles ROC, ROD, are equiangular, whence ROD = RCO, or ACO. Therefore ROD = AOB the difference of the angles ROA, ROB; and RD is the tangent of that difference.

Now by the firſt analogy, $AC = \frac{AOq}{AB} = \frac{ROq + RAq}{RB - RA}$. And by the ſecond analogy, $RD = \frac{ROq}{RC}$, that is, $= \frac{ROq}{AC + RA} = \frac{ROq}{\frac{ROq + RAq}{RB - RA} + RA} = \frac{ROq}{\frac{ROq + RB \times RA}{RB - RA}} = \frac{ROq \times \overline{RB - RA}}{\overline{ROq + RB \times RA}}$. Whence (multiplying by $\overline{ROq + RB \times RA}$) $RD \times \overline{ROq + RB \times RA} = ROq \times \overline{RB - RA}$; and $RD : RB - RA :: ROq : ROq + RB \times RA$. QED.

82. Corol. 1. Hence the tangent of the ſum of any number of given angles may be eaſily computed. Put $RO = r$, $RA = a$, and $RB = b$.

then the tangent of the ſum of the angles whoſe tangents are a and b, that is, RD, is $= \frac{b+a}{rr-ab} \times rr$.

Call this tangent x. Then for the ſame reaſon the tangent of the ſum of this laſt angle, and a third angle whoſe tangent is c, will be $\frac{x+c}{rr-xc} \times rr$, or (by ſubſtituting the value of x in it's ſtead) $\frac{rr \times \overline{a+b+c}-abc}{rr-ab-ac-bc}$, = the tangent of the ſum of the three angles whoſe tangents are a, b, c. And ſo on.

The ſubſtitution of the value of x is thus performed. Firſt find the value of the numerator $\overline{x+c} \times rr$, of the fraction, and then of the denominator. Now the numerator $\overline{x+c} \times rr$ $= \overline{\frac{brr+arr}{rr-ab}+c} \times rr =$ $\frac{brr+arr+crr-abc}{rr-ab} \times rr$. And $rr-xc$ $= rr \frac{-bcrr-acrr}{rr-ab} = \frac{r^4-abrr-bcrr-acrr}{rr-ab}$.

Therefore $\frac{\overline{x+c} \times rr}{rr-xc} =$ $\frac{\overline{brr+arr+crr-abc} \times rr}{r^4-abrr-bcrr-acrr} =$ $\frac{rr \times \overline{a+b+c}-abc}{rr-ab-bc-ac}$.

83. COROL. 2. And hence may the tangent of any multiple of a given angle be computed. For 'tis only making $a=b=c$; and for the tangent of a double angle we ſhall have $\Big(\frac{b+a}{rr-ab}$

$\times\, rr =)\ \frac{2a}{rr-aa} \times rr$. And for the tangent of a treble angle $\frac{3arr-a^3}{rr-3aa}$. And ſo on.

This propoſition and demonſtration is taken from *Smith*'s Optics. The ſubſtance of it is demonſtrated in another manner by Mr. *Machin* in a ſolution he has given of the following problem.

PROBLEM.

84. The tangents of two arcs being given, to find the tangent of their ſum.

Let AB, GE (*Fig.* 39.) be the given tangents. AD is the tangent ſought.

Put CA $= r$
AB $= a$
GE $= b$
AD $= c$. Draw EL parallel to AD. The triangles ABC, EFG, are ſimilar to one another each being ſimilar to CFL. Therefore CA : CB :: EG : EF, or $r : \sqrt{rr+aa} : b$: EF. Whence EF $= \frac{b \times \sqrt{rr+aa}}{r}$.

Again, CA : AB :: EG : GF, or $r : a :: b$: GF. Therefore GF $= \frac{ab}{r}$.

But CF $=$ CG $-$ GF $= r - \frac{ab}{r} = \frac{rr-ab}{r}$.

And from the ſimilarity of the triangles CBD, CFE, CF : EF :: CB : BD; that is, $\frac{rr-ab}{r} :: \frac{b\sqrt{rr \times aa}}{r} :: \sqrt{rr+aa}$: BD. Whence BD

$=b$

$=\frac{brr+baa}{rr-ab}$. Therefore $c\left(=a+\frac{brr+baa}{rr-ab}\right)=\frac{rra-baa+brr+baa}{rr-ab}=\frac{\overline{b+a}\times rr}{rr-ab}$. QED.

COROL. I. Hence may be deduced Dr. *Smith*'s proportions for finding the tangents of the ſum and difference. For firſt, $\frac{rr\times\overline{a+b}}{rr-ab} : \overline{a+b} :: rr : rr-ab$. Which is the firſt part. And here, as before, if $a=b$, we ſhall have $c\left(=\frac{rr\times\overline{a+b}}{rr-ab}\right)=\frac{2arr}{rr-aa}$.

Secondly, Let c be the greater tangent, a the leſſer, and b the tangent ſought. Then ſince $c=\frac{rr\times\overline{b+a}}{rr-ab}$, or $crr-abc=brr+arr$, it follows that $crr-arr=brr+bac$, whence $b=\frac{rr\times\overline{c-a}}{rr+ac}$, or $rr : rr+ac :: \frac{rr\times\overline{c-a}}{rr+ac}$, or $b : c-a$.

Scholium.

85. In the firſt part of Dr. *Smith*'s propoſition a difficulty may perhaps ariſe with reſpect to the fourth term in the proportion, to wit, the ſquare of the radius diminiſhed by the rectangle under the tangents; which, it may be thought, may ſometimes be a negative quantity, namely, when the ſquare of the radius is leſs than the rectangle under the tangents. In anſwer to this we muſt

obſerve that this can never happen while the ſum of the angles does not exceed a right angle. For if the two angles are together exactly equal to a right angle, the radius will be a mean proportional between their two tangents (by *P*. 2, *Corol.* 3.) and of conſequence the ſquare of the radius will be equal to the rectangle contained under them. In this caſe therefore the fourth term is *o*; and ſo it ought to be, becauſe the tangent of the ſum, which is the firſt term, is the tangent of a right angle, and therefore infinite, and conſequently bears an infinitely great ratio of majority to the ſum of the tangents; therefore that the ſquare of the radius may bear the ſame proportion to the fourth term, that term muſt be nothing. Now let one of theſe angles, whoſe ſum is equal to a right angle, be diminiſhed; it's tangent will be diminiſhed likewiſe. Conſequently the rectangle under the tangents will be leſs than the ſquare of the radius.

But Dr. *Smith*'s conſtruction is general, and will ſerve for determining the tangents of the ſum and difference of any two angles, when that ſum or difference exceeds a right angle; as we now proceed to ſhew diſtinctly in the following manner. Firſt, let the angle AOB (*Fig.* 40.) be greater than a right angle; it's tangent will be the ſame with that of it's ſupplement to two right angles. Make AB : AO :: AO : AC, and RC : RO :: RO : RD. And we ſhall have ACO=AOB; and ROD = RCO, that is, to the ſupplement of ACO, or AOB to two right angles. Therefore RD is the tangent of the ſum AOB of the angles ROA, ROB. But RD $= \frac{ROq}{RC} =$

ROq

$$\frac{ROq}{RA-AC} = \frac{ROq}{RA-\frac{AOq}{AB}} =$$

$$\frac{ROq}{RA-\frac{ROq-RAq}{RB+RA}} = \frac{ROq}{\frac{RB\times RA-ROq}{RB+RA}} =$$

$\frac{ROq\times\overline{RB+RA}}{RB\times RA-ROq}$. Whence RD : RB+RA :: ROq : RB×RA —ROq; or the tangent of the ſum is to the ſum of the tangents as the ſquare of the radius to the rectangle under the tangents diminiſhed by the ſquare of the radius.

Secondly, Let the ſum of the angles ROA, ROB, or ROG, ROK (*Fig.* 41.) be greater than a right angle, but their difference AOB leſs than a right angle; the conſtruction of the figure being the ſame as before, RD will be the tangent of the difference, and is found in exactly the ſame manner, and by the ſame proportion as when the ſum of the angles is leſs than a right angle, as in Part 2d. of the demonſtration, *Fig.* 33.

Thirdly, Let not only the ſum of the angles ROG, ROK, but likewiſe their difference BOK (*Fig.* 42.) be greater than a right angle. RA is the tangent of ROK; and the tangent of BOK is the ſame with that of AOB. The rectangle under RB, RA is leſs than the ſquare of the radius, becauſe AOB is leſs than a right angle. This caſe therefore falls under Part 1, of the demonſtration, *Fig.* 32, and is determined by ſaying, as the ſquare of the radius diminiſhed by the rectangle under the tangents is to the ſquare of the radius, ſo is the ſum of the tangents to the tangent of the ſum of the leſſer of the two angles, and the ſupplement of the greater to two right angles; or to the tangent of the difference of the angles.

86. If any one is deſirous of ſeeing theſe caſes made to come under the very words of the propoſition, it may be done by conſidering the tangent of an obtuſe angle as negative. For then in the laſt equation for *Fig.* 40. *viz.* $RD = \frac{ROq \times \overline{RB+RA}}{RB \times \overline{RA-ROq}}$, we muſt ſubſtitute $-RD$ inſtead of RD, becauſe the angle AOB is obtuſe; therefore the other ſide of the equation muſt be made negative, by changing the ſigns of the members of the denominator; and the reſulting equation will be $-RD = \frac{ROq \times \overline{RB+RA}}{ROq - RB \times RA}$; whence $-RD : RB+RA :: ROq : ROq - RB \times RA$, agreeably to the words of the propoſition.

In like manner, in *Fig.* 42, RA is negative; conſequently the rectangle under RB, RA is negative; and the proportion for determining the tangent of AOB, to wit, $ROq - RB \times RA : ROq :: RB+RA : RD$ will be expreſſed thus; $ROq + RB \times RA : ROq :: RB - RA : RD$, as it is in the propoſition.

See *Saunderſon*'s Algebra, *Art.* 319.

87. Having in the ſecond corollary to the laſt propoſition ſhewn how, when the tangent of a circular arch, and the radius of the circle, are given, the tangent of any multiple of that arch may be thence computed, it will not be amiſs to do the ſame thing for the ſines and chords of the multiples of a given arch, or angle; particularly as this may be ſo eaſily performed by the application of a few of the propoſitions that have been already

Fig. 28.

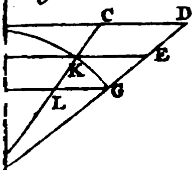


Fig. 29.

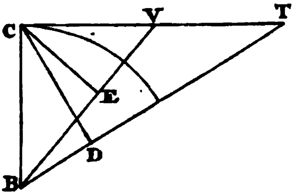
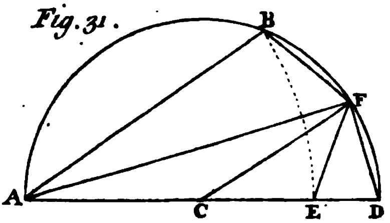


Fig. 31.



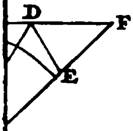


Fig. 34.

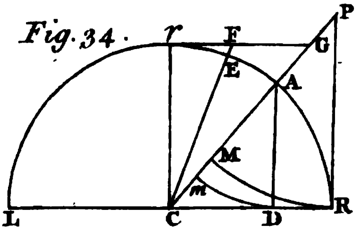
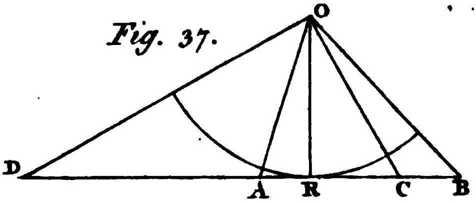


Fig. 37.



J. Mynde sc.

the triple arc LB being leſs than a quadrant, the ſingle arc LO will be leſs than 30°, and conſequently r will be greater than $2s$, and *a fortiori* rr than $2ss$. Therefore CR is $= \frac{r^2 - 2s^2}{r}$. But by *Prop.* 6, we have CL : 2CR :: OD : BA — OD, or $r : \frac{2r^2 - 4s^2}{r} :: s : x - s$; therefore $x - s$ is $= \frac{2rrs - 4s^3}{rr}$, and x, or BA the ſine of the triple arc LB, is $= \frac{3rrs - 4s^3}{rr}$ QEI.

Secondly, Since the arcs LO, LB, LS are an arithmetical progreſſion whereof the common difference is OB, or LM, it follows, by *Prop.* 6, that CL : 2CA :: MR : SP — OD. But becauſe BA has been found to be $= \frac{3rrs - 4s^3}{rr}$, BAq will be $= \frac{9r^4s^2 - 24r^2s^4 16s^6}{r^4}$, and conſequently CA$q$ ($=$ CBq — BAq) $= rr$

$$\frac{-9r^4s^2 + 24r^2s^4 - 16s^6}{r^4} =$$

$$\frac{r^6 - 9r^4s^2 + 24r^2s^4 - 16s^6}{r^4} =$$

$\frac{r^4 - 8r^2s^2 + 16s^4}{r^4} \times \overline{rr - ss}$, the affirmative root of which is $\frac{r^2 - 4s^2}{rr} \times \sqrt{\overline{rr - ss}}$, becauſe, the arc LS being ſuppoſed to be leſs than a quadrant, it's fifth part LO muſt be much leſs than 30°, and conſequently r muſt be greater than $2s$, and rr than $4ss$. Therefore CA is $= \frac{r^2 - 4s^2}{rr}$

$\times \sqrt{rr - ss}$, and 2CA $= \frac{2r^2 - 8s^2}{rr} \times \sqrt{rr - ss}$. But MR was before found to be $= \frac{2s}{r} \times \sqrt{rr - ss}$. Therefore $r : \frac{2r^2 - 8s^2}{rr} \times \sqrt{rr - ss} :: \frac{2s}{r} \times \sqrt{rr - ss} : x - s$; conſequently $x - s$ is $= \frac{4r^2 s - 16s^3}{r^4} \times \overline{rr - ss} = \frac{4r^4 s - 20r^2 s^3 + 16s^5}{r^4}$, and x, or SP the ſine of the quintuple arc LS, is $= \frac{5r^4 s - 20r^2 s^3 + 16s^5}{r^4}$ QEI.

Thirdly, Since the arcs LB, LS, LG, are an arithmetical progreſſion whereof the common difference is BS, or LM, it follows, by *Prop.* 6, that LC : 2CP :: MR : GF — BA. But MR has been ſhewn to be $= \frac{2s}{r} \times \sqrt{rr - ss}$, and BA to be $= \frac{3rrs - 4s^3}{rr}$; and in the caſe of the quintuple arc we found PS to be $= \frac{5r^4 s - 20r^2 s^3 + 16s^5}{r^4}$; therefore PS$q$ is $= \frac{25r^8 s^2 - 200r^6 s^4 + 560r^4 s^6 - 640r^2 s^8 + 256s^{10}}{r^8}$; conſequently CP$q$ (= CSq — PSq) $= \frac{r^{10} - 25r^8 s^2 + 200r^6 s^4 - 560r^4 s^6 + 640r^2 s^8 - 256s^{10}}{r^8}$ $= \frac{r^8 - 24r^6 s^2 + 176r^4 s^4 - 384r^2 s^6 + 256s^8}{r^8} \times \overline{rr - ss}$, the affirmative ſquare root of which is $\frac{r^4 - 12r^2 s^2 + 16s^4}{r^4} \times \sqrt{rr - ss}$, as may be thus demonſtrated. This quantity, 'tis evident, will be an affirmative quantity if it can be ſhewn that $r^4 - 12r^2s^2 + 16s^4$ is an affirmative quantity, or that $12r^2s^2$ is leſs than $r^4 + 16s^4$, or that $\frac{12r^2s^2}{16}$ is leſs than $\frac{r^4}{16} + s^4$, or $\frac{3r^2s^2}{4}$ than $\frac{4r^4}{64} + s^4$. Now if $\frac{3r^2s^2}{4}$ were leſs than s^4, as would be the caſe if s^2 were greater than $\frac{3r^2}{4}$, and conſequently s than $\frac{\sqrt{3}}{2} \times r$, or than the ſine of 60°, the quantity $\frac{3r^2s^2}{4}$ would a *fortiori* be leſs than $s^4 + \frac{4r^4}{16}$. But becauſe the greateſt magnitude of the arch LO, of which s is the ſine, is in the preſent caſe the ſeventh part of the quadrant, or $\frac{90^\circ}{7}$, or about 12°, 51′, 25″, 'tis evident that $\frac{3r^2s^2}{4}$ will be greater than s^4. We are therefore to prove that, though $\frac{3r^2s^2}{4}$ is greater than s^4, yet it is leſs in the preſent caſe than $s^4 + \frac{4r^4}{64}$, or, which is the ſame thing, that $\frac{3r^2s^2}{4} - s^4$ is in the preſent caſe leſs than $\frac{4r^4}{64}$. By *El.* 2. 5, it appears that the quantity $\frac{3r^2s^2}{4} - s^4$ will encreaſe continually from o to $\frac{9r^4}{64}$ while ss increaſes from o to $\frac{3rr}{8}$, and will decreaſe again continually from $\frac{9r^4}{64}$ to o while ss increaſes further from $\frac{3rr}{8}$ to $\frac{3rr}{4}$, and conſequently will at two different inſtants of time, the one before ss being equal to $\frac{3rr}{8}$, the other while ss is of an intermediate magnitude between $\frac{3rr}{8}$ and $\frac{3rr}{4}$, be equal (to any given quantity that is leſs than $\frac{9r^4}{64}$, and conſequently) to $\frac{4r^4}{64}$. Therefore while ss is leſs than the leſſer of theſe two values (as alſo afterwards when it is greater than the greater of theſe two values, but leſs than $\frac{3rr}{4}$ which is it's greateſt magnitude) the quantity $\frac{3r^2s^2}{4} - s^4$ will be leſs than $\frac{4r^4}{64}$. Now when $\frac{3r^2s^2}{64} - s^4$ is $=$ $\frac{4r^4}{64}$ the leſſer value of ss is $.095492 \times rr$, and conſequently the leſſer value of s muſt be $.309 \times r =$ S, 18°: For, ſubtracting both ſides of the equation $\frac{3r^2s^2}{4} - s^4 = \frac{4r^4}{64}$ from $\frac{9r^4}{64}$, we have $\frac{9r^4}{64} - \frac{3r^2s^2}{4} + s^4 = \frac{5r^4}{64}$, and

than

conſequently (the leſſer value of ss being leſs than $\frac{3rr}{8}$) $\frac{3rr}{8} - ss = \frac{\sqrt{5}}{8} \times rr$, and $ss = \frac{3-\sqrt{5}}{8} \times rr = \frac{3-2.23606}{8} \times rr = \frac{.76394}{8} \times rr = .095492 \times rr$, and $s = .309 \times r =$ S, 18°, 00′. But in the preſent caſe the arch LO can never be greater than 12°, 51′, 25″, and conſequently the ſine OD, or s, muſt be leſs than the leſſer of the two values of s in the equation $\frac{3r^2s^2}{4} - s^4 = \frac{4r^4}{64}$, and OD$q$, or ss, than the leſſer of the two values of ss in that equation; therefore the quantity $\frac{3r^2s^2}{4} - s^4$ is in the preſent caſe leſs than $\frac{4r^4}{64}$, or $\frac{r^4}{16}$, and $r^4 - 12r^2s^2 + 16s^4$ is an affirmative quantity. Therefore CP is $= \frac{r^4 - 12r^2s^2 + 16s^4}{r^4} \times \sqrt{rr - ss}$, and $r : \frac{2r^4 - 24r^2s^2 + 32s^4}{r^4} \times \sqrt{rr - ss} : \frac{2s}{r} \times \sqrt{rr - ss} : x \quad \frac{-3rrs + 4s^3}{rr}$; whence x $\frac{-3rrs + 4s^3}{rr} = \frac{4r^4s - 48r^2s^3 + 64s^5}{r^6} \times \overline{rr - ss} = \frac{4r^6s - 52r^4s^3 + 112r^2s^5 - 64s^7}{r^6}$, and conſequently x, or GF, the ſine of the ſeptuple arch LG, is $= \frac{7r^6s - 56r^4s^3 + 112r^2s^5 - 64s^7}{r^6}$. QEI.

Fourthly, Since the arcs LS, LG, LQ are in arithmetical progreſſion, whereof the common difference is SG, or LM, 'tis evident we might diſcover QZ by this proportion, LC : 2CF :: MR : QZ — SP; and in the ſame manner may the ſine of the 11th, 13th, 15th, 17th, 19th, and of every following odd multiple of LO, be diſcovered, to wit, by conſidering it and the two preceding odd multiples of LO as an arithmetical progreſſion whereof the common difference is LM or 2LO. But where the arc whoſe ſine is to be found happens to be an odd multiple of ſome other of the preceding odd multiples of LO as well as of LO itſelf, it's ſine may be determined in an eaſier manner by means of the equation belonging to that preceding arc: thus, in the preſent inſtance, LQ being triple of LB, we may determine QZ by the help of the equation BA = $\frac{3rrs - 4s^3}{rr}$; and this method of finding it being ſomewhat eaſier than the former, in which the computation of the value of CF is pretty laborious, we ſhall here make uſe of it. Since BA is = $\frac{3rrs - 4s^3}{rr}$, it follows that, if we put y = BA, we ſhall have QZ, or x, = $\frac{3rry - 4y^3}{rr}$. But y being equal to $\frac{3rrs - 4s^3}{rr}$, y^3 will be equal to $\frac{27r^6s^3 - 108r^4s^5 + 144r^2s^7 - 64s^9}{r^6}$; therefore x, or QZ, the ſine of the noncuple arc LQ, is $\left(= \frac{3rry - 4y^3}{rr}\right) = \frac{9rrs - 12s^3}{rr}$

$$\frac{-108r^{6}s^{3} + {}^{4}432rs^{5} - 576r^{2}s^{7} + 256s^{9}}{r^{8}}$$

$$= \frac{9r^{8}s - 120r^{6}s^{3} + 432r^{4}s^{5} - 576r^{2}s^{7} + 256s^{9}}{r^{8}}$$

QEI.

89. In this manner may the ſines of all the odd multiples of the arc LO be found. As to the even multiples, they, 'tis evident, are all of them double, either of the odd multiples, or of inferior even multiples, which are themſelves either double, or double of the double, or double of the double of the double, &c. that is, either double, or quadruple, or octuple, &c. of the odd multiples; and conſequently their ſines may be found from thoſe of the odd multiples by *Prop.* 4, in the manner following.

In the firſt place, the ſine MR of the double arc LM has been already found to be $= \frac{2s}{r} \times \sqrt{rr - ss}$.

Secondly, Since LK the quadruple arc is double of LM, it follows, by *Prop.* 4, that LC : CR : : 2MR : KN, that is, is, $r : \frac{rr - 2ss}{r} :: \frac{4s}{r} \times \sqrt{rr - ss} : x$; whence x, or NK, the ſine of the quadruple arc LK, is $= \frac{4r^{2}s - 8s^{3}}{r^{3}} \times \sqrt{rr - ss}$.

Thirdly, Since LT the ſextuple arc is double of LB the triple arc, it follows, by *Prop.* 4, that LC:CA::2BA:TV, that is, (becauſe BA has been already found to be equal to $\frac{3rrs - 4s^{3}}{rr}$, and CA to be $= \frac{rr - 4ss}{rr} \times \sqrt{rr - ss}$) $r : \frac{rr - 4ss}{rr} \times \sqrt{rr}$

$\sqrt{rr - ss} :: \frac{6rr - 8s^{3}}{rr} : x$; whence x, or TV, the ſine of the ſextuple arc LT, is $= \frac{6r^4 s - 32r^2 s^3 + 32s^5}{r^5} \times \sqrt{rr - ss}$.

Fourthly, Since LH the octuple arc is double of LK, it follows, by *Prop.* 4, that LC : CN :: 2KN : HI. But it was juſt now ſhewn that KN is $= \frac{4r^2 s - 8s^3}{r^3} \times \sqrt{rr - ss}$; therefore 2KN is $= \frac{8r^2 s - 16s^3}{r^3} \times \sqrt{rr - ss}$, KN$q$ $= \frac{16r^4 s^2 - 64r^2 s^4 + 64s^6}{r^6} \times \overline{rr - ss} = \frac{16r^6 s^2 - 80r^4 s^4 + 128r^2 s^6 - 64s^8}{r^6}$, and CN$q$ (= CKq — KNq) $= \frac{r^8 - 16r^6 s^2 + 80r^4 s^4 - 128r^2 s^6 + 64s^8}{r^6}$. The affirmative ſquare root of this quantity is $\frac{r^4 - 8r^2 s^2 + 8s^4}{r^3}$; as may be thus demonſtrated: 'Tis evident $\frac{r^4 - 8r^2 s^2 + 8s^4}{r^3}$ will be an affirmative quantity if $8r^2 s^2$ is leſs than $r^4 + 8s^4$, or $8r^2 s^2 - 8s^4$ than r^4, or $r^2 s^2 - s^4$ than $\frac{r^4}{8}$. Now it appears by *El.* 2. 5, that the quantity $r^2 s^2 - s^4$ will increaſe continually from o to $\frac{r^4}{4}$ while ss increaſes from o to $\frac{rr}{4}$, and will decreaſe again continually from $\frac{r^4}{4}$ to o while ss

increaſes further from $\frac{rr}{2}$ to rr, and conſequently will at two different inſtants of time, the one before ss becomes equal to $\frac{rr}{2}$, and the other while ss is of an intermediate magnitude between $\frac{rr}{2}$ and rr, be equal (to any given quantity that is leſs than $\frac{r^4}{4}$, and conſequently) to $\frac{r^4}{8}$. Therefore while ss is leſs than the leſſer of theſe two values (as alſo afterwards when it is greater than the greater of theſe two values, but leſs than rr, which is it's greateſt magnitude) the quantity $r^2s^2-s^4$ will be leſs than $\frac{r^4}{8}$. Now when $r^2s^2-s^4$ is $=\frac{r^4}{8}$, the leſſer value of ss is $=.146446,\times rr$, and conſequently the leſſer value of s is $=.3826\times r=$ S, 22°, 30′; for ſubtracting both ſides of the equation $r^2s^2-s^4=\frac{r^4}{8}$ from $\frac{r^4}{4}$, we ſhall have $\frac{r^4}{4}-r^2s^2+s^4$ $\left(=\frac{r^4}{4}-\frac{r^4}{8}\right)=\frac{r^4}{8}$, and conſequently (becauſe the leſſer value of ss is leſs than $\frac{rr}{2}$) $\frac{rr}{2}-ss=\frac{rr}{\sqrt{8}}$, and $ss=\frac{rr}{2}-\frac{rr}{\sqrt{8}}=\overline{\frac{1}{2}-\frac{1}{\sqrt{8}}}\times rr=\frac{\sqrt{8}-2}{2\sqrt{8}}\times rr=\frac{2.828427-2}{2\times2.828427}\times rr$ $=\frac{.828427}{5.656854}\times rr=.146446\times rr$, and conſequently $s = .3826 \times r =$ S, 22°, 30. But in the preſent caſe the arch LO, whereof s is the ſine, can never be greater than the eighth part of a quadrant, or $\frac{90°}{8}$, or 11°, 15′; conſequently the ſine OD, or s, muſt be leſs than the leſſer of the two values of s in the equation $r^2s^2 - s^4 = \frac{r^4}{8}$, and OD$q$, or ss, than the leſſer of the two values of ss in that equation; therefore the quantity $r^2s^2 - s^4$ is in the preſent caſe leſs than $\frac{r^4}{8}$, and $\frac{r^4 - 8r^2s^2 + 8s^4}{r^3}$ is an affirmative quantity. Therefore CN is $= \frac{r^4 - 8r^2s^2 + 8s^4}{r^3}$, and conſequently $r : \frac{r^4 - 8r^2s^2 + 8s^4}{r^3} :: \frac{8r^2s - 16s^3}{r^3} \times \sqrt{rr - ss} : x$; whence x, or HI, the ſine of the octuple arc LH, is $= \frac{8r^6s - 80r^4s^3 + 192r^2s^5 - 128s^7}{r^7} \times \sqrt{rr - ss}$. And in the ſame manner may the ſines of all the following even multiples of the arc LO be computed. QEI.

90. COROL. I. Hence we may deduce the following concluſions, to wit.

Firſt, That, ſince, when x repreſents the ſine of the double arc, $\frac{2s}{r} \times \sqrt{rr - ss}$ is $= x$, we ſhall in that caſe have $\frac{4ss}{rr} \times \overline{rr - ss} = xx$ and $4r^2s^2 - 4s^4 = r^2x^2$.

Secondly,

Secondly, That, ſince, when x repreſents the ſine of the triple arc, $\frac{3rrs - 4s^3}{rr}$ is $= x$, we ſhall in that caſe have $3rrs - 4s^3 = rrx$.

Thirdly, That, ſince, when x repreſents the ſine of the quadruple arc, $\frac{4r^2s - 8s^3}{r^3} \times \sqrt{rr - ss}$ is $= x$, we ſhall in that caſe have $\frac{16r^4s^2 - 64r^2s^4 + 64s^6}{r^6} \times \overline{rr - ss}$, or $\frac{16r^6s^2 - 80r^4s^4 + 128r^2s^6 - 64s^8}{r^6} = xx$, and $16r^6s^2 - 80r^4s^4 + 128r^2s^6 - 64s^8 = r^6x^2$.

Fourthly, That, ſince, when x repreſents the ſine of the quintuple arc, $\frac{5r^4s - 20r^2s^3 + 16s^5}{r^4}$ is $= x$, we ſhall in that caſe have $5r^4s - 20r^2s^3 + 16s^5 = r^4x$.

Fifthly, That, ſince, when x repreſents the ſine of the ſextuple arc, $\frac{6r^4s - 32r^2s^3 + 32s^5}{r^5} + \sqrt{rr - ss}$ is $= x$, we ſhall in that caſe have $\frac{36r^8s^2 - 384r^6s^4 + 1408r^4s^6 - 2048r^2s^8 + 1024s^{10}}{r^{10}} \times \overline{rr - ss}$, or $\frac{36r^{10}s^2 - 420r^8s^4 + 1792r^6s^6 - 3456r^4s^8 + 3072r^2s^{10} - 1024s^{12}}{r^{10}} = xx$, and $36r^{10}s^2 - 420r^8s^4 + 1792r^6s^6 - 3456r^4s^8 + 3072r^2s^{10} - 1024s^{12} = r^{10}x^2$.

Sixthly, That, ſince, when x repreſents the ſine of the ſeptuple arch, $\frac{7r^6s - 56r^4s^3 + 112r^2s^5 - 64s^7}{r^6}$ is $= x$, we ſhall in that caſe have $7r^6s - 56r^4s^3 + 112r^2s^5 - 64s^7 = r^6x$.

Seventhly, That, ſince, when x repreſents the ſine of the octuple arch, it is equal to $\frac{8r^6s - 80r^4s^3 + 192r^2s^5 - 128s^7}{r^7} \times \sqrt{rr - ss}$, we ſhall in that caſe have $\frac{64r^{12}s^2 - 1280r^{10}s^4 + 9472r^8s^6 - 32768r^6s^8 + 57344r^4s^{10} - 49152r^2s^{12} + 16384s^{14}}{r^{14}} \times rr - ss$, or $\frac{64r^{14}s^2 - 1344r^{12}s^4 + 10752r^{10}s^6 - 42240r^8s^8 + 90112r^6s^{10} - 106496r^4s^{12} + 65536r^2s^{12} - 16384s^{16}}{r^{14}}$ is $= xx$, and $64r^{14}s^2 - 1344r^{12}s^4 + 10752r^{10}s^6 - 42240r^8s^8 + 90112r^6s^{10} - 106496r^4s^{12} + 65536r^2s^{14} - 16384s^{16} = r^{14}x^2$.

And *Eighthly*, That, ſince, when x repreſents the ſine of the noncuple arc, it is $= \frac{9r^8s - 120r^6s^3 + 432r^4s^5 - 576r^2s^7 + 256s^9}{r^8}$ we ſhall in that caſe have $9r^8s - 120r^6s^3 + 432r^4s^5 - 576r^2s^7 + 256s^9 = r^8x$.

Therefore the equations expreſſing the relations of the ſines of the double, triple, quadruple, quintuple, ſextuple, ſeptuple, octuple, and noncuple arcs, to the ſine of the ſimple arc, are as follows.

For the double arc, $4r^2s^2 - 4s^4 = r^2x^2$.

For the triple arc, $3r^2s - 4s^3 = r^2x$.

For the quadruple arc, $16r^6s^2 - 80r^4s^4 + 128r^2s^6 - 64s^8 = r^6x^2$.

For the quintuple arc, $5r^4s - 20r^2s^3 + 16s^5 = r^4x$.

For the ſextuple arc, $36r^{10}s^2 - 420r^8s^4 + 1792r^6s^6 - 3456r^4s^8 + 3072r^2s^{10} - 1024s^{12} = r^{10}x^2$.

For the ſeptuple arc, $7r^6s - 56r^4s^3 + 112r^2s^5 - 64s^7 = r^6x$.

For the octuple arc, $64r^{14}s^2 - 1344r^{12}s^4$ +

$+ 10752r^{10}s^6 - 42240r^8s^8 + 90112r^6s^{10} - 106496r^4s^{12} + 65536r^2s^{14} - 16384s^{16} = r^{14}x^2$.
And for the noncuple arc, $9r^8s - 120r^6s^3 + 432r^4s^5 - 576r^2s^7 + 256s^9 = r^8 x$.

91. Corol. 2. Hence 'tis evident if the ſine x of any circular arch that is not greater than a quadrant, and the radius of the circle, be given, the ſine s of the half, third, fourth, fifth, or any other given part of that arch may be found by the reſolution of one of the foregoing equations, $4r^2s^2 - 4s^4 = r^2x^2$, $3r^2s - 4s^3 = rrx$, &c. obtained in the laſt corollary. But of theſe equations, 'tis evident, thoſe that relate to the ſines of the even multiples of the arch LO riſe to a higher number of dimenſions than thoſe that relate to the odd multiples of the ſame arch; the number of dimenſions to which the latter equations riſe being always equal to the number of parts into which the arc is to be divided, whereas the number of dimenſions to which the former equations riſe is always double the number of parts into which the arc is to be divided. And hence it is, as I imagine, that Sir *Iſaac Newton* in the 29th problem of his Algebra choſe to determine the multiples and parts of a given angle, or circular arch, by their coſines and the chords of their complements, rather than by their ſines and chords, becauſe (as ſhall preſently be ſhewn) the number of dimenſions to which the equations that determine thoſe lines riſe, is never greater than, but always equal to, the number of parts into which the arc is to be divided.

92. COROL. 3. It will be worth our while, for the ſake of ſhewing the conformity between theſe determinations of the ſines of multiple arcs and thoſe which may be obtained by means of an infinite ſeries hereafter to be delivered, to ſet down the values of the ſines of the even multiples which all of them involve in them the irrational quantity $\sqrt{rr - ss}$, in infinite ſerieſes of ſimple terms, as follows. If we extract the ſquare root of $rr - ss$ in the manner deſcribed by Dr. *Saunderſon* in his Algebra, *Art.* 19, we ſhall find it to be $= r \frac{-s^2}{2r} \frac{-s^4}{8r^3} \frac{-s^6}{16r^5} \frac{-5s^8}{128r^7} \frac{-7s^{10}}{256r^9}$ &c. *ad infinitum.* It follows therefore in the firſt place that the ſine of the double arc (being $= \frac{2s}{r} \times \sqrt{rr - ss}$) is $= 2s \frac{-s^3}{rr} \frac{-s^5}{4r^4} \frac{-s^7}{8r^6} \frac{-5s^9}{64r^8} \frac{-7s^{11}}{128r^{10}}$, &c. *ad infinitum.*

Secondly, That the ſine of the quadruple arc (being $= \frac{4r^2s - 8s^3}{r^3} \times \sqrt{rr - ss}$) is $= 4s - \frac{10s^3}{r^2} + \frac{7s^5}{2r^4} + \frac{3s^7}{4r^6} + \frac{11s^9}{32r^8} + \frac{13s^{11}}{64r^{10}}$ +, *&c.*

Thirdly, That the ſine of the ſextuple arc (being $= \frac{6r^4s - 32r^2s^3 + 32s^5}{r^5} \times \sqrt{rr - ss}$) is $= 6s \frac{-35s^3}{r^2} \frac{+189s^5}{4r^4} \frac{-99s^7}{8r^6} \frac{-143s^9}{64r^8} \frac{-117s^{11}}{128r^{10}}$ −, *&c.*

And *Fourthly*, That the ſine of the octuple arc (being $= \frac{8r^6s - 80r^4s^3 + 192r^2s^5 - 128s^7}{r^7}$ $\times \sqrt{rr - ss}$) is $= 8s \frac{-84s^3}{r^2} \frac{+231s^5}{r^4}$ $\frac{-429s^7}{2r^6} \frac{+715s^9}{16r^8} \frac{+221s^{11}}{32r^{10}} +$, &c.

93. Corol. 4. Hitherto we have ſuppoſed the greateſt, or multiple, arc not to exceed a quadrant: But before we quit this ſubject, it will be proper to examine how far theſe equations are true, and what alterations they undergo, when the greateſt arc exceeds a quadrant, and is of any magnitude leſs than the whole circle. Theſe caſes we ſhall find are determined in the following manner: "When the greater arc is "greater than a quadrant, but leſs than a ſemi- "circle, the value of x is expreſt by the very ſame "equations as when it is leſs than a quadrant; "and when it is greater than a ſemicircle, but "leſs than a whole circle, by the ſame equations "with the ſigns of the terms on their left hand "ſides every where changed: Or, in other words, "the value of x is equal to the difference of the "ſame terms in both caſes; but thoſe terms "which in the former caſe were negative, or "which were leſs than the others, and therefore "ſubtracted from them, do in the latter caſe "become the greater, ſo that the others muſt "now be ſubtracted from them, and marked "with the negative ſign."

94. In order to prove this it will be neceſſary to conſider farther the fourth and ſixth propoſitions, which have hitherto been ſuppoſed to re-

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ble only to caſes wherein the greateſt arc does not exceed a quadrant, and to inquire how far they are true when that reſtriction is taken away.

95. The firſt of theſe propoſitions is true univerſally, whatever be the magnitude of the greateſt arc; to wit, that as the radius of a circle is to the coſine of any arc, ſo is twice the ſine of that arc to the ſine of it's double.

This will appear by the inſpection of *Fig.* 3, 44, 45, 46; in all which BL is the ſingle arc, BD the double arc, BM and CM the ſine and coſine of BL, and DE the ſine of BD. For from the ſimilarity of the triangles CBM, BDE, we have $CB : CM :: DB = 2BM : DE$.

96. The ſecond of theſe propoſitions is true whenever the three arcs in arithmetical proportion are either all leſs or all greater than a ſemicircle. But when the greateſt exceeds, and the leaſt arc falls ſhort of a ſemicircle, it diſcovers the ſum, inſtead of the difference, of the ſines of the extreme arcs.

This will appear by the inſpection of *Fig.* 4, 47, 48, 49, 50, 51, 52, 53, 54, 55; in all which BE, BD, BF are the three arcs in arithmetical proportion; EL, DK, FI, their ſines; CD a radius biſecting the chord EF in O; and OM, EG, lines drawn parallel to the diameter BCN, meeting FI (produced, if need be) in M and G. For from the ſimilarity of the triangles CDK, FOM, we have $CD : CK :: FO : FM$, and $CD : 2CK :: FO : 2FM = FG$; and 'tis evident, FG is the difference of FI and EL, when F and E lie on the ſame ſide of the diameter BCN, and their ſum, when F and E lie on the different ſides of that diameter.

97. By theſe propoſitions we will firſt demonſtrate the aſſertion in *Art.* 93, in the caſes of the triple, quintuple, and ſeptuple arcs, ſeparately; and afterwards endeavour to point out a general reaſon why it muſt be true in all other caſes whatſoever.

98. Firſt then, I ſay, that if the triple arc be not greater than a ſemicircle, the relation between x and s will be expreſt by this equation, $x = \frac{3rrs - 4s^3}{rr}$, or $3rrs - 4s^3 = rrx$; if it be greater, by this equation, $x = \frac{4s^3 - 3rrs}{rr}$, or $4s^3 - 3rrs = rrx$.

The greateſt magnitude of the ſimple arc, or that which it has when the triple arc is equal to a whole circle, being 120°, and conſequently leſs than a ſemicircle, it follows by the laſt article that when the triple arc is greater than a ſemicircle, but leſs than a whole circle, the proportion of *Prop.* 6, will diſcover the ſum, and not the difference, of the ſines of the ſimple and triple arcs. Further, becauſe by *Art.* 19, the ſine of the double arc is always $= \frac{2s}{r} \times \sqrt{rr - ss}$ and conſequently it's ſquare to $\frac{4r^2s^2 + 4s^4}{rr}$, it follows that the coſine of the double arc is always equal to the ſquare root of the quantity $\frac{r^4 - 4r^2s^2 + 4s^4}{rr}$: But $\frac{r^2 - 2s^2}{r}$ is the ſquare root of this quantity as long as $2s^2$ is leſs than r^2, or the ſimple arch is leſs than 45°, and $\frac{2s^2 - r^2}{r}$

is it's ſquare root when $2s^2$ is greater than r^2, or, s greater than the ſine of 45°, and therefore as long as the ſimple arc is of any magnitude leſs than 135°; therefore when the ſimple arc is leſs than 45°, or the triple arc than 135°, the coſine of the double arc is $= \frac{r^2 - 2s^2}{r}$, and when the ſimple arc is greater than 45°, but leſs than 120°, or the triple arc is greater than 135°, but leſs than the whole circle, the coſine of the double arc is $= \frac{2s^2 - r^2}{r}$. But at the inſtant $2s^2$ begins to be greater than r^2, or the coſine of the double arc begins to be $= \frac{2s^2 - r^2}{r}$, the ſine x of the triple arc begins to be leſs than s the ſine of the ſimple arc; becauſe at that inſtant the triple arc, being equal to 135°, exceeds a quadrant by exactly the ſame quantity as the ſimple arc, which is 45°, falls ſhort of it, and conſequently the ſines of the triple and ſimple arc are equal; therefore when the triple arc is of any magnitude greater than 135°, but leſs than a ſemircle, (and conſequently it's ſine is leſs than the ſine of 135°, and the ſine of the ſimple arc, which will now be nearer to a quadrant than before, is greater than the ſine of 45°) the ſine of the triple arc will be leſs than the ſine of the ſimple arc, or x will be leſs than s. The proceſſes therefore are as follows.

When the triple arc is leſs than 135°, we have $r : \frac{2r^2 - 4s^2}{r} :: s : x - s$; therefore $x - s$ is $= \frac{2r^2s - 4s^3}{rr}$, and $x = \frac{3r^2s - 4s^3}{rr}$.

When the triple arc is between 135° and 180, we have $r : \frac{4s^2 - 2r^2}{r} :: s : s - x$; therefore $s - x$ is $= \frac{4s^3 - 2r^2 s}{rr}$, and $x = \frac{3r^2 s - 4s^3}{rr}$, as before.

Laſtly, when the triple are is greater than a ſemicircle, but leſs than a whole circle; we have $r : \frac{4s^2 - 2r^2}{r} :: s : s + x$; therefore $s + x$ is $= \frac{4s^3 - 2r^2 s}{rr}$, and $x = \frac{4s^3 - 3r^2 s}{rr}$.

99. *Secondly*, If the quintuple arc be not greater than a ſemicircle, the relation of x and s will be expreſt by this equation, $x = \frac{5r^4 s - 20r^2 s^3 + 16s^5}{r^4}$, or $5r^4 s - 20r^2 s^3 + 16s^5 = r^4 x$; and if the quintuple arc is greater than a ſemicircle, but leſs than a whole circle, by this equation, $x = \frac{20r^2 s^3 - 5r^4 s - 16s^5}{r^4}$, or $20 r^2 s^3 - 5r^4 s - 16s^5 = r^4 x$.

For the ſingle, triple, and quintuple arcs are an arithmetical progreſſion, of which the common difference is the double arc, whoſe ſine is $\frac{2s}{r} \times \sqrt{rr - ss}$. And the greateſt magnitude of the ſimple arc, or that which it has when the quintuple arc is equal to a whole circle, being only the fifth part of the circle, is much leſs than a ſemicircle: Conſequently by *Art.* 96, when the quintuple arc is greater than a ſemicircle, but leſs than a whole circle, the proportion of *Prop.* 6, will diſcover the ſum, and not the difference, of the ſines of the ſimple and quintuple arcs.

Further, the ſine of the mean, or triple, arc has been ſhewn to be either $\frac{3rrs - 4s^3}{rr}$ or $\frac{4s^3 - 3rrs}{rr}$, according as the triple arc is leſs or greater than a ſemicircle; therefore in either caſe the ſquare of the ſine of the mean arc is $= \frac{9r^4s^2 - 24r^2s^4 + 16s^6}{r^4}$, and conſequently the ſquare of the coſine of the mean arc (being equal to the exceſs of rr, or the ſquare of the radius, above the ſquare of the ſine) is $= \frac{r^6 - 9r^4s^2 + 24r^2s^4 - 16s^6}{r^4} = \frac{r^4 - 8r^2s^2 + 16s^4}{r^4} \times \overline{rr - ss}$, of which $\frac{r^2 - 4s^2}{r^2} \times \sqrt{rr - ss}$ is the affirmative root when the ſingle arc does not exceed 30°, and $\frac{4s^2 - r^2}{r^2} \times \sqrt{rr - ss}$ when the ſingle arc is of any magnitude between 30°, and 150°; therefore when the quintuple arc is leſs than 150°, or the ſimple arc than 30°, the coſine of the triple, or mean, arc is $= \frac{r^2 - 4s^2}{r^2} \times \sqrt{rr - ss}$, and when the quintuple arc is of any magnitude greater than 150°, but leſs than the whole circle, or the ſimple arc is of any magnitude greater than 30°, but leſs than $\frac{360}{5}$, or 72°, the coſine of the mean, or triple, arc is $= \frac{4s^2 - r^2}{r^2} \times \sqrt{rr - ss}$. Moreover, when the quintuple arc is leſs than 150°, or the ſingle arc than 30°, the ſine x is greater than s; but afterwards, when the quintuple arc is between 150° and 180°, the ſine x is leſs than s.

The proceſſes therefore are as follows.

When the quintuple arc is under 150°, we have $r : \frac{2r^2 - 8s^2}{r^2} \times \sqrt{rr - ss} :: \frac{2s}{r} \times \sqrt{rr - ss} : x - s$; therefore $x - s$ is $= \frac{4r^2 s - 16s^3}{r^4} \times \overline{rr - ss} = \frac{4r^4 s - 20r^2 s^3 + 16s^5}{r^4}$, whence $x = \frac{5r^4 s - 20r^2 s^3 + 16s^5}{r^4}$.

When the quintuple arc lies between 150° and 180°, we have $r : \frac{8s^2 - 2r^2}{r^2} \times \sqrt{rr - ss} :: \frac{2s}{r} \times \sqrt{rr - ss} : s - x$; therefore $s - x$ is $= \frac{16s^3 - 4r^2 s}{r^4} \times \overline{rr - ss} = \frac{20r^2 s^3 - 4r^4 s - 16s^5}{r^4}$; whence x is $= \frac{5r^4 s - 20r^2 s^3 + 16s^5}{r^4}$, as before.

Laſtly, When the quintuple arc is of any magnitude greater than a ſemicircle, but leſs than a whole circle, we have $r : \frac{8s^2 - 2r^2}{r^2} \times \sqrt{rr - ss} :: \frac{2s}{r} \times \sqrt{rr - ss} : x + s$; therefore $x + s$ is $= \frac{20r^2 s^3 - 4r^4 s - 16s^5}{r^4}$, and x

$=$

$= \dfrac{20r^2s^3 - 5r^4s - 16s^5}{r^4}$.

100. *Thirdly*, If the ſeptuple arc be leſs than a ſemicircle, the relation of x and s will be expreſt by this equation, $x = \dfrac{7r^6s - 56r^4s^3 + 112r^2s^5 - 64s^7}{r^6}$, or $7r^6 - 56r^4s^3 + 112r^2s^5 - 64s^7 = r^6x$; and if the ſeptuple arc is greater than a ſemicircle, but leſs than a whole circle, by this equation, $x = \dfrac{64s^7 - 112r^2s^5 + 56r^4s^3 - 7r^6s}{r^6}$; or $64s^7 - 112r^2s^5 + 56r^4s^3 - 7r^6s = r^6x$.

For the triple, quintuple, and ſeptuple arcs are an arithmetical progreſſion, of which the common difference is the double arc, whoſe ſine is $\dfrac{2s}{r} \times \sqrt{rr - ss}$. And the triple arc muſt in this caſe be neceſſarily leſs than a ſemicircle, becauſe it's greateſt magnitude, or that which it has when the ſeptuple arc is equal to a whole circle, is only three ſevenths of the whole circle. Therefore, when the ſeptuple arc is greater than a ſemicircle, but leſs than a whole circle, it follows by *Art.* 96, that the proportion of *Prop.* 6, will diſcover the ſum, and not the difference, of the ſines of the triple and ſeptuple arcs: Alſo (by what has been already ſhewn in the caſe of the triple arc) the ſine of the triple arc will be $= \dfrac{3rrs - 4s^3}{rr}$. Further, the ſquare of the coſine of the quintuple, or mean, arc, being equal to the ſquare of the radius diminiſhed by the ſquare of it's ſine, will be $= rr -$ the ſquare of $\frac{5r^4s - 20r^2s^3 + 16s^5}{r^4}$, or of

$\frac{20r^2s^3 - 5r^4s - 16s^5}{r^4}$, (for both thoſe quantities have the ſame ſquare), that is, to

$$\frac{r^{10} - 25r^8s^2 + 200r^6s^4 - 560r^4s^6 + 640r^2s^8 - 256s^{10}}{r^8}$$

$$= \frac{r^8 - 24r^6s^2 + 176r^4s^4 - 384r^2s^6 + 256s^8}{r^8}$$

$\times \overline{rr - ss}$, of which $\frac{r^4 - 12r^2s^2 + 16s^4}{r^4} \times \sqrt{rr - ss}$ is the affirmative root as long as the ſingle arc (as may eaſily be deduced from the ſame article) is leſs than 18°, (as was ſhewn in Art. 7.) and afterwards $\frac{12r^2s^2 - r^4 - 16s^4}{r^4} \times \sqrt{rr - ss}$, when the ſingle arc is between 18° and 54°. Therefore as long as the ſimple arc is leſs than 18°, or the ſeptuple arc than 126°, the coſine of the quintuple, or mean, arc is $= \frac{r^4 - 12r^2s^2 + 16s^4}{r^4} \times \sqrt{rr - ss}$, and when the ſimple arc is greater than 18°, but leſs than 51°, 25′, 42″, or the ſeptuple arc is greater than 126°, but leſs than a whole circle, the coſine of the mean arc is $\frac{12r^2s^2 - r^4 - 16s^4}{r^4} \times \sqrt{rr - ss}$.

Moreover, when the ſeptuple arc is leſs than 126°, or the ſimple arc than 18°, the ſine x of the ſeptuple arc will be greater than the ſine of the triple arc, and when the ſeptuple arc is greater than 126°, but leſs than a ſemicircle, the ſine x of the ſeptuple arc will be leſs than the ſine of the triple arc; for at the inſtant the ſeptuple arc is equal to 126°, the quintuple arc is equal to 90°, or a quadrant, and therefore the triple and ſeptuple arcs differ equally from a quadrant, whence 'tis evident their ſines will at that time be equal, and conſequently as the ſeptuple arc approaches nearer to a ſemicircle, and the triple arc to a quadrant, the ſine of the ſeptuple arc will become leſs than the ſine of the triple arc. The proceſſes therefore are as follows.

When the ſeptuple arc is leſs than 126°, we have $r : \frac{2r^4 - 24r^2s^2 + 32s^4}{r^4} \times \sqrt{\overline{rr-ss}} :: \frac{2s}{r} \times \sqrt{\overline{rr-ss}} : x \frac{-3rrs + 4s^3}{rr}$; therefore $x \frac{-3rrs + 4s^3}{rr}$ is $=$ $\frac{4r^4s - 48r^2s^3 + 64s^5}{r^6} \times \overline{rr-ss} =$ $\frac{4r^6s - 52r^4s^3 + 112r^2s^5 - 64s^7}{r^6}$, whence x is $= \frac{7r^6 - 56r^4s^3 + 112r^2s^5 - 64s^7}{r^6}$

When the ſeptuple arc is greater than 126°, but leſs than a ſemicircle, we have $r : \frac{24r^2s^2 - 2r^4 - 32s^4}{r^4} \times \sqrt{\overline{rr-ss}} :: \frac{2s}{r} \times \sqrt{\overline{rr-ss}} : \frac{3rrs - 4s^3}{rr} - x$; therefore $\frac{3rrs - 4s^3}{rr} - x$ is $= \frac{48r^2s^3 - 4r^4s - 64s^5}{r^6} \times \overline{rr-ss} = \frac{64s^7 - 112r^2s^5 + 52r^4s^3 - 4r^6s}{r^6}$, whence x is $= \frac{7r^6s - 56r^4s^3 + 112r^2s^5 - 64s^7}{r^6}$;

as before.

Laſtly, When the ſeptuple arc is greater than a ſemicircle, but leſs than a whole circle, we have $r : \frac{24r^2s^2 - 2r^4 - 32s^4}{r^4} \times \sqrt{rr - ss} :: \frac{2s}{r} \times \sqrt{rr - ss} : x \frac{+3rrs - 4s^3}{rr}$; therefore $x \frac{+3rrs - 4s^3}{rr}$ is $= \frac{48r^2s^3 - 4r^4s - 64s^5}{r^6} \times \overline{rr - ss} = \frac{64s^7 - 112r^2s^5 + 52r^4s^3 - 4r^6}{r^6}$, whence x is $= \frac{64s^7 - 112r^2s^5 + 56r^4s^3 - 7r^6s}{r^6}$.

101. After an attentive conſideration of the three preceding caſes, it will, I imagine, be ſufficiently evident to every one, that the reaſon of this regular change in the value of x depends upon this obſervation; "that at the very ſame "inſtant of time that the coſine of the mean arc "changes, the ſine of the greater extreme be"comes leſs than the ſine of the leſſer extreme; "and that, in all the foregoing inſtances, when "once the coſine of the mean arc is changed, "it continues ſo ever after, ſo long as the ex"treme arc is of any magnitude leſs than the "whole circle." This has been already proved concerning each of the foregoing equations ſeparately: We will therefore now endeavour to give a general reaſon why this muſt be true in all of them, and afterwards, ſhew that a ſimilar reaſon will produce the ſame change in the ſigns of the terms in all other caſes whatſoever.

102. And firſt; the ſine of the greater extreme begins to be leſs than the ſine of the

leſſer extreme at the inſtant the mean arc is equal to a quadrant; becauſe, the extreme arcs differing at that inſtant equally from a quadrant, their ſines will be exactly equal. Afterwards, when the mean arc is greater than a quadrant, and the greater extreme, though ſtill leſs than a ſemicircle, approaches nearer to it than it did before, it's ſine will be leſs than the ſine of the leſſer extreme. And the coſine of the mean arc changes likewiſe at the inſtant the mean arc is equal to a quadrant; becauſe at that time it is preciſely equal to nothing, or thoſe parts of it's value (which, 'tis evident is always expreſt in powers and multiples of r and s) which were negative, or which before were leſs than the others, and therefore ſubtracted from them, are now by the increaſe of s made to equal them: Whence it follows (as might be demonſtrated from the nature of the increaſe of the terms, as we ſhall ſhew in a ſimilar inſtance in the 9th corollary of this propoſition, *Art.* 133.) that if the mean arc be greater than a quadrant, or s be ſtill farther increaſed, thoſe parts will become the greater, and the others muſt be ſubtracted from them. This is ſometimes expreſt thus; “a quantity that continually decreaſes till it “becomes equal to nothing, and afterwards “increaſes again, becomes after ſuch increaſe “a negative quantity, or will have paſt from “affirmation through nothing into negation.” But as expreſſions of this kind have ſomewhat of an obſcure myſterious air in them, we have endeavoured as much as poſſible to avoid them.

Secondly, When once the coſine of the mean arc is changed, it remains ſo ever after in all the foregoing inſtances, as long as the greater extreme arc is of any magnitude leſs than the whole circle. For as s increaſes or decreaſes continually, or by infinitely ſmall degrees, ſo as never to paſs from one degree of magnitude to another without paſſing through all the intermediate degrees of magnitude thy lie between them, 'tis evident that the terms of which the value of the coſine of the mean arc conſiſts, and which involve the ſine s in them, muſt vary alſo by infinitely ſmall degrees: Conſequently when the negative terms, or thoſe which were at firſt the leſſer, become firſt equal to, and afterwards greater than, the affirmative terms, they cannot again become leſs than the affirmative terms without firſt becoming a ſecond time equal to them; that is, the negative terms cannot become a ſecond time leſs than the affirmative terms before the coſine of the mean arc vaniſhes, or becomes equal to 0, a ſecond time, and therefore not in any of the caſes that have hitherto been explained; for the coſine of an arc does not, as is evident, vaniſh a ſecond time 'till the arc becomes equal to three quadrants, and in the foregoing inſtances the mean arc can never be ſo great as three quadrants, becauſe even in the caſe of the ſeptuple arc (in which the mean arc is greater than in any of the inferiour caſes) the greateſt magnitude of the mean arc, or that which it has when the ſeptuple arc is equal to the whole circle, is $\frac{5}{7}$ or $\frac{20}{28}$ of the whole circumference, which is leſs than three quadrants, or $\frac{3}{4}$ or $\frac{21}{28}$ of the whole circumference.

103. In the following odd multiples above the ſeptuple arc, the mean arc may be greater than three quadrants: Thus, for inſtance, in the caſe of the noncuple arc, in which the three equidifferent arcs are the quintuple, ſeptuple, and noncuple arcs, 'tis evident the greateſt magnitude of the ſeptuple, or mean, arc, or that which it has when the noncuple arc is equal to a whole circle, is $\frac{7}{9}$, or $\frac{28}{36}$, of the whole circumference, which is greater than $\frac{27}{36}$ or $\frac{3}{4}$ of the whole circumference, or three quadrants; and ſince this is true in the noncuple arc, it follows a *fortiori* that it will be true in all higher multiples. But in all theſe caſes it muſt be obſerved firſt, that when the mean arc is greater than three quadrants, the leſſer extreme arc muſt be greater than a ſemicircle, and conſequently by *Art.* 96, the proportion of *Prop.* 6, will diſcover the difference, and not any more the ſum of the ſines of the extreme arcs: For the greater extreme arc being ſuppoſed never to exceed a whole circle, it follows that when the mean arc is greater than three quadrants, the exceſs of the greater extreme arc above it muſt be leſs than a quadrant; therefore the exceſs of the mean arc above the leſſer extreme arc muſt alſo be leſs than a quadrant, and conſequently the leſſer extreme arc muſt be greater than a ſemicircle, and by *Art.* 96, the proportion of *Prop.* 6, will diſcover the difference, and not the ſum, of the ſines of the extreme arcs. *Secondly*, we muſt obſerve that at the inſtant the mean arc becomes equal to three quadrants, the ſines of the

two extreme arcs will become equal to each other, (becauſe at that inſtant the two extreme arcs will differ equally from three quadrants), and therefore, when the mean arc is greater than three quadrants, the ſine of the greater extreme arc will be leſs than the ſine of the leſſer extreme arc; but when the mean arc is leſs than three quadrants, and the leſſer extreme arc is greater than a ſemicircle, the ſine of the greater extreme arc will be greater than the ſine of the leſſer extreme arc. *Thirdly*, it muſt be obſerved that the ſigns of the terms of the value of the ſine of the quintuple arc, to wit, the quantity $\frac{5r^4s - 20r^2s^3 + 16s^5}{r^4}$, are contrary, when the quintuple arc is greater than a ſemicircle, but leſs than a whole circle, to what they were when the quintuple arc was leſs than a ſemicircle, as was demonſtrated in the caſe of the quintuple arc; that is, in the caſe of the noncuple arc, the ſigns of the terms that compoſe the value of the ſine of the leſſer extreme of the three equidifferent arcs (which are in this caſe the quintuple, the ſeptuple, and the noncuple arcs), to wit, the quintuple arc become contrary, when the leſſer extreme, or quintuple arc is greater than a ſemicircle, to what they were when the leſſer extreme, or quintuple arc was leſs than a ſemicircle. Therefore in the caſe of the noncuple arc, the terms of the value of the ſine of the leſſer extreme arc change their ſigns at the ſame time that the fourth term of the proportion of *Prop.* 6, from being equal to the ſum of the ſines of the extreme arcs, becomes equal to the exceſs of the ſine of the greater extreme arc above the ſine of the leſſer extreme arc, the three other terms of that proportion continuing the ſame as before; and theſe two changes compenſate each other, and make the ſigns of the terms of the value of the ſine of the greater extreme, or noncuple arc, the ſame as where the leſſer extreme arc was leſs than a ſemicircle: And when the mean arc becomes greater than three quadrants, and the terms of it's coſine change their ſines, and conſequently the terms of the ſecond quantity of the proportion of *Prop.* 6, (which is double of the coſine of the mean arc) change their ſines, the fourth term of that proportion changes it's ſines likewiſe, and from being equal to the exceſs of the ſine of the greater extreme arc above the ſine of the leſſer extreme arc, becomes equal to the exceſs of the ſine of the leſſer extreme arc above the ſine of the greater extreme arc, the firſt and third terms of that proportion continuing ſtill the ſame as before; and theſe two changes compenſate each other, and make the ſines of the terms of the ſine of the greater extreme arc the ſame as in the former caſe. Therefore when the noncuple arc is of any magnitude between a ſemicircle and a whole circle, the ſines of the terms of the value of it's ſine are contrary to what they are when that arc is leſs than a ſemicircle. And from the reaſoning contained in this proof 'tis evident that if the ſines of the terms that compoſe the value of the ſines of multiple arcs change in the manner here deſcribed in one inſtance, they will likewiſe change in the ſame manner in all higher inſtances: Conſequently ſince it has been ſhewn that the ſigns of the terms that compoſe the value of the ſine of the noncuple arc do change in this manner, it follows that the ſigns of the terms that compoſe the values of the ſines of all higher multiple arcs do alſo change in the ſame manner.

104. The whole of the preceding reaſoning upon theſe changes of the ſigns may be briefly expreſſed in Algebraic characters in the following manner. Put x for the ſine of the multiple, or greater extreme arc, y for the ſine of the leſſer extreme arc, r for the radius of the circle, s for the ſine of the ſimple arc, and $a-b$ for the coſine of the mean arc expreſt in powers of r and s; and let $\frac{2s}{r} \times \sqrt{rr-ss}$ be put $= e$. Then 'twill be evident in the firſt place, that when the greater extreme arc is leſs than a ſemicircle, and the mean arc is leſs than a quadrant, x will be greater than y, and $x-y$ will be the fourth term in the proportion of *Prop.* 6; alſo the coſine of the mean arc, not having yet changed it's ſigns, will be $a-b$. Therefore in this caſe the proportion of *Prop.* 6, will be as follows, $r : 2 \times \overline{a-b} :: e : x-y$, whence $x-y$ is $= \frac{2e}{r} \times \overline{a-b}$, and $x = \frac{2e}{r} \times \overline{a-b} + y$.

Secondly, When the greater extreme arc is leſs than a ſemicircle, and the mean arc is greater than a quadrant, y will be greater than x, and the fourth term in the proportion of *Prop.* 6, will be $y-x$; and in this caſe the coſine of the mean arc, having changed it's ſigns, will be $b-a$. Therefore in this caſe we have $r : 2 \times \overline{b-a} :: e : y-x$, whence $y-x$ is $= \frac{2e}{r} \times \overline{b-a}$

and $x = \frac{2e}{r} \times \overline{a-b} + y$.

Thirdly, When the greater extreme arc is greater than a ſemicircle, but the leſſer extreme arc is leſs than a ſemicircle, the fourth term in the proportion of *Prop.* 6, will be $x+y$, and therefore we ſhall have $r : 2 \times \overline{b-a} :: e : x+y$, whence $x+y$ is $= \frac{2e}{r} \times \overline{b-a}$, and $x = \frac{2e}{r} \times \overline{b-a} - y$.

Fourthly, When both the greater and the leſſer extreme arcs are greater than a ſemicircle, but the mean arc is leſs than three quadrants, x will be greater than y, and the fourth term in the proportion of *Prop.* 6, will be $x-y$; therefore in this caſe we ſhall have $r : 2 \times \overline{b-a} :: e : x-y$, whence $x-y$ is $= \frac{2e}{r} \times \overline{b-a}$, and $x = \frac{2e}{r} \times \overline{b-a} + y$.

Fifthly, When the mean arc is greater than three quadrants, it's coſine will be changed a ſecond time, and will therefore be $a-b$ as at firſt, and y will be greater than x, and conſequently the fourth term of the proportion of *Prop.* 6, will be $y-x$. Therefore $r : 2 \times \overline{a-b} :: e : y-x$, whence $y-x$ is $= \frac{2e}{r} \times \overline{a-b}$, and $x = \frac{2e}{r} \times \overline{b-a} + y$.

Now if $c-d$ be ſuppoſed to repreſent the value of y when expreſt in powers and multiples of r and s, c being equal to the ſum of all the affirmative terms of that value when the leſſer extreme arc is leſs than a ſemicircle, and *d* to the ſum of all the negative terms of that value, and it be ſhewn in the caſe of the leſſer extreme arc (as it has been ſhewn in the caſes of the triple, and quintuple arcs) that when the leſſer extreme arc is greater than a ſemicircle, the ſigns of the terms that compoſe it's ſine are contrary to what they are when that arc is leſs than a ſemicircle, 'tis evident that, when that arc is greater than a ſemicircle, *d* will be greater than *c*, and y will be $= d - c$. Therefore in the three firſt of the foregoing caſes y will be $= c - d$, and in the two laſt to $d - c$. Subſtitute therefore $c - d$ inſtead of y in the three firſt values of x, and $d - c$ inſtead of y in the two laſt values of x, and we ſhall have

In the *Firſt* place, $x = \frac{2e}{r} \times \overline{a-b} + y = \frac{2e}{r} \times \overline{a-b} + c - d$.

Secondly, $x = \frac{2e}{r} \times \overline{a-b} + y = \frac{2e}{r} \times \overline{a-b} + c - d$.

Thirdly, $x = \frac{2e}{r} \times \overline{b-a} - y = \frac{2e}{r} \times \overline{b-a} - c + d$.

Fourthly, $x = \frac{2e}{r} \times \overline{b-a} + y = \frac{2e}{r} \times \overline{b-a} - c + d$. And,

Fifthly, $x = \frac{2e}{r} \times \overline{b-a} + y = \frac{2e}{r} \times \overline{b-a} - c + d$. That is, in the two firſt caſes, in which the greater extreme, or multiple arc is leſs than a ſemicircle, it's ſine x will be equal to the quantity $\frac{2e}{r} \times \overline{a-b+c-d}$, and in the three latter caſes, in which the greater extreme, or multiple arc is greater than a ſemicircle, but leſs than a whole circle, it's ſine x will be equal to the quantity $\frac{2e}{r} \times \overline{b-a-c+d}$, which conſiſts of the ſame terms as the former value of x, only the ſines are every where changed.

105. 'Tis evident that all that has been here ſaid relates to even multiples of a given arc as well as to odd ones, ſince their ſines may be found by *Prop.* 6, as well as the ſines of the odd multiples; to wit, by making the two preceding even multiples of the ſimple arc the firſt and ſecond terms of an arithmetical progreſſion, of which the multiple arc whoſe ſine is to be inveſtigated is the third term, and the double arc the common difference.

106. If any one is deſirous of ſeeing the two determinations of the value of x in *Art.* 93, reduced to one, it may be done by conſidering the ſine of an arc greater than a ſemicircle as negative. For then by ſubſtituting $-x$ inſtead of x in the equation exhibiting the ſecond value of x, and changing the ſigns of the terms of that value to make it negative, it will be converted into the ſame quantity as the former value. Thus, if in the equation $x = \frac{4s^3 - 3rrs}{rr}$, which determines the value of x, when the triple arc is greater than a ſemicircle, we ſubſtitute $-x$ inſtead of x, and make the other ſide of the equa-

tion negative by changing the ſines of the terms, we ſhall have $-x = \frac{3rrs - 4s^3}{rr}$, the ſame quantity as when the triple arc was leſs than a ſemicircle.

107. It will not perhaps be amiſs, before we put an end to this corollary, to obſerve in two or three inſtances taken from the equation for the triple arc, how perfectly the values of s determined by it coincide with thoſe before found in the ſame caſes in Part firſt.

When the ſimple arc is half a quadrant, it's ſine is equal to the ſine of the triple arc, and the equation $3rrs - 4s^3 = rrx$ becomes $3rrs - 4s^3 = rrs$, or $2rrs - 4s^3 = 0$; whence $rr = 2ss$, as it ought to be by *El.* 1. 47, becauſe at that time the ſine and coſine are equal.

When the ſimple arc is 60°, the triple arc is a ſemicircle, and $x = 0$. Therefore the equation becomes in this caſe $3rrs - 4s^3 = 0$; whence $rr : ss :: 4 : 3 :: 1 : \frac{3}{4}$, and $r : s :: 1 : \frac{\sqrt{3}}{2}$, as in *Prop.* 2.

When the ſimple arc is a quadrant, s and x become each of them equal to the radius, and the equation $4s^3 - 3rrs = rrx$ becomes $4r^3 - 3r^3 = r^3$, which proves itſelf.

When the ſimple arc is 120°, the triple arc is equal to the whole circle, and it's ſine is 0. Therefore the equation $4s^3 - 3rrs = rrx$ becomes in this caſe $4s^3 - 3rrs = 0$; whence $r : s :: 1 : \frac{\sqrt{3}}{2}$, as it ought to be, becauſe s is equal to the ſine of 60°.

Fig. 39.

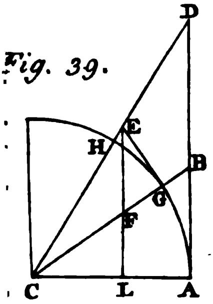
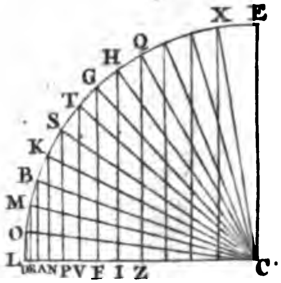
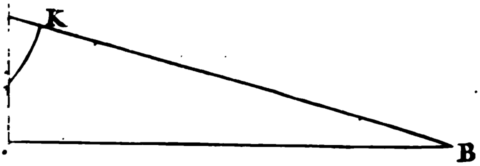


Fig. 43.





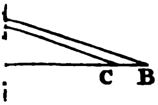


Fig. 44.

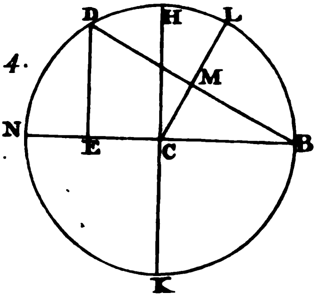
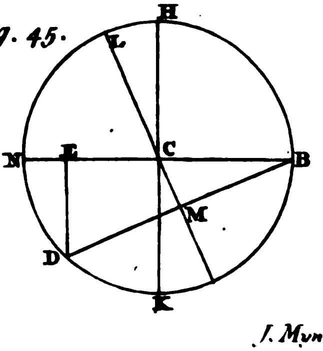


Fig. 45.



J. Mynde. fc.

108. COROL. 5. The coſines of the multiples of a given arc, or their relations to the coſine of the given arc itſelf, may be determined by the help of *Coroll.* 4, and their values, when the multiple arc is greater than one quadrant, but leſs than three quadrants, will be expreſt by the ſame quantities as when it is leſs than one quadrant, or greater than three quadrants; only the ſines of the terms muſt every where be changed.

For let x be the coſine of the ſimple arc, and q of the multiple arc. Then will $xx = rr - ss$, and $x = \sqrt{rr - ss}$.

And for the double arc we ſhall have $q = \frac{r^2 - 2s^2}{r} = \frac{rr - 2rr + 2xx}{r} = \frac{2xx - rr}{r}$, as long as the double arc is leſs than a quadrant, and alſo when it is greater than three quadrants.

And $q = \frac{2ss - rr}{r} = \frac{2rr - 2xx - rr}{r} = \frac{rr - 2xx}{r}$, when the double arc is greater than one quadrant, but leſs than three quadrants.

For the triple arc, when either leſs than one quadrant, or greater than three quadrants, we have $q = \frac{r^2 - 4s^2}{r^2} \times \sqrt{rr - ss} = \frac{rr - 4rr + 4xx}{rr} \times x = \frac{4x^3 - 3rrx}{rr}$; when greater than one quadrant, but leſs than three quadrants, $q = \frac{4s^2 - r^2}{rr} \times \sqrt{rr - ss} = \frac{3rrx - 4x^3}{rr}$.

For the quadruple arc; q is equal to the ſquare root of the difference of rr and the ſquare of

110. Corol. 6. From the foregoing corollary it follows that the equations that expreſs the relations of the coſine x of any circular arc to the coſine q of it's double, triple, quadruple, and quintuple, are, when the multiple arch, whereof q is the coſine, is either not greater than one quadrant, or greater than three quadrants, theſe that follow, to wit.

For the double arc, $2xx - rr = q$,

For the triple arc, $4x^3 - 3rrx = rrq$,

For the quadruple arc, $8x^4 - 8r + r^4 = r^3q$,

$-8r^2x^2$

And for the quintuple arc, $16x^5 - 20r^2x^3 + 5r^4x = r^4q$.

And when the multiple arc, whereof q is the coſine, is greater than one quadrant, but leſs than three quadrants, the equations of the coſines are theſe, to wit.

For the double arc, $rr - 2xx = rq$,

For the triple arc, $3rrx - 4x^3 = rrq$,

For the quadruple arc, $8r^2x^2 - 8x^4 - r^4 = r^3q$,

And for the quintuple arc, $20r^2x^3 - 16x^5 - 5r^4x = r^4q$.

And by the reſolution of one of theſe equations, 'tis evident, we may find the coſine x of the half, third, fourth, or fifth part, of any circular arch whatſoever whoſe coſine q, together with the radius r of the circle, is given.

111. Corol. 7. As to Sir *Iſaac Newton*'s method of finding theſe equations, it may be explained as follows. Let the given angle, of whoſe multiples we are to find the ſines, be FAG (*Fig.* 56.) Inſcribe in this angle the lines AB, BC, CD, DE, &c. of any given length whatſoever

47.

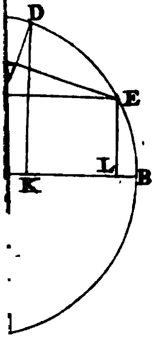
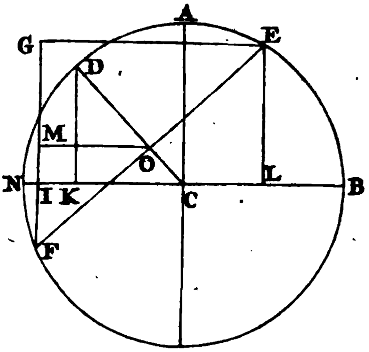


Fig. 48.



g. 50.

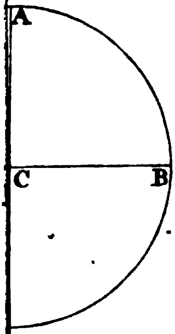
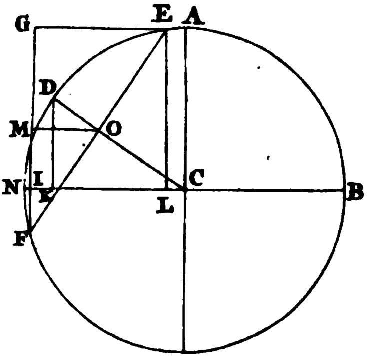


Fig. 51.



53.

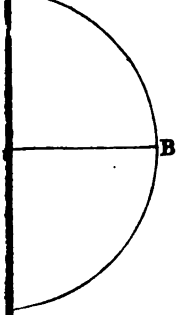
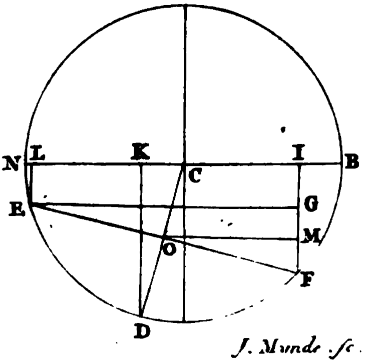


Fig. 54.



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ſoever, ſo that they ſhall all be equal to each other. Then 'tis evident that as the triangles ABC, BCD, CDE, EDF, *&c.* are iſoſceles, the angles at the baſe of each of them will be equal. Therefore $CBD = A + ACB = 2A$; $DCE = A + ADC = A + CBD = 3A$; $EDF = A + AED = A + DCE = 4A$; and $FEG = A + AFE = A + EDF = 5A$; and ſo on: Wherefore, drawing BK, CL, DM, EN, *&c.* perpendicular to AC, BD, CE, DF, *&c.* reſpectively, 'tis evident AK, BL, CM, DN, *&c.* will be the coſines, and BK, CL, DM, EN, *&c.* the ſines, of the angles A, 2A, 3A, 4A, *&c.* in circles whoſe radii are the equal lines AB, BC, CD, DE, *&c.* Put $AB = r$, and AK (the coſine of the given angle) $= x$; and from the ſimilarity of the triangles ABK, ACL, ADM, AEN, *&c.* we ſhall have the following proportions.

In the firſt place we have $AB : AK :: AC : AL$, or $r : x :: 2x : AL$; whence AL is $= \frac{2xx}{r}$, and conſequently BL, the coſine of twice the given angle, is $(= AL - AB) = \frac{2xx}{r} - r$, or $\frac{2xx - rr}{r}$:

Secondly, We have $AB : AK :: AD : AM$, or (becauſe AD is $= AL + LD = AL + LB = 2AL - AB$) $AB : AK :: 2AL - AB : AM$; that is, $r : x :: \frac{4xx}{r} - r : AM$; therefore AM is $= \frac{4x^3}{rr} - x$, and CM, the coſine of three times the given angle, is (= AM —

AC = AM — 2AK) $= \frac{4x^3}{rr} - 3x$, or $\frac{4x^3 - 3rrx}{rr}$.

Thirdly, We have AB : AK :: AE (= 2AM — AC) : AN; that is, $r : x :: \frac{8x^3}{rr} - 4x$: AN; therefore AN is $= \frac{8x^4}{r^3} - \frac{4xx}{r}$, and DN, the coſine of four times the given angle, is (= AN — AD $= \frac{8x^4}{r^3} - \frac{4xx}{r} - \frac{4xx}{r} + r$) $= \frac{8x^4}{r^3} - \frac{8x^2}{r} + r$, or $\frac{8x^4 - 8r^2x^2 + r^4}{r^3}$.

Fourthly, We have AB : AK :: AF (= 2AN — AD) : AO; that is, $r : x ::$ ($\frac{16x^4}{r^3} - \frac{8xx}{r} - \frac{4xx}{r} + r =$) $\frac{16x^4}{r^3} - \frac{12xx}{r} + r$: AO; therefore AO is $= \frac{16x^5}{r^4} - \frac{12x^3}{rr} + x$, and EO, the coſine of five times the given angle, is (= AO — AE $= \frac{16x^5}{r^4} - \frac{12x^3}{rr} + x - \frac{8x^3}{rr} + 4x$) $= \frac{16x^5}{r^4} - \frac{20x^3}{rr} + 5x$, or $\frac{16x^5 - 20r^2x^3 + 5r^4x}{r^4}$. In the ſame manner we may find the coſines of greater multiples.

112. Hence 'tis evident that if q be put ſucceſſively for the coſines of the double, triple, quadruple, and quintuple arcs, the equations expreſſing the relations of theſe coſines to the

coſine x of the ſimple arc FAG, will be theſe that follow, to wit,

For the double arc, $2xx - rr = rq$,

For the triple arc, $4x^3 - 3rrx = rrq$,

For the quadruple arc, $8x^4 - 8r^2x^2 + r^4 = r^3q$,

And for the quintuple arc, $16x^5 - 20r^2x^3 + 5r^4x = r^4q$, which are exactly the ſame equations as thoſe obtained in *Corol.* 6.

113. Theſe equations, 'tis evident, are true as long as the coſine q has any magnitude, or as long as the arc ſubtending the given multiple angle is of any magnitude leſs than a quadrant; for ſo long the radii DE, EF, FG, *&c.* will be greater than the coſines DM, EN, FO, *&c.* which are the perpendicular, or leaſt, diſtances of the lines AF, AG, that contain the angle FAG; and therefore ſo long may thoſe radii be inſcribed in that angle, and the conſtruction of *Fig.* 56, and the reaſonings founded upon it will remain the ſame: Which agrees with the determinations in *Corol.* 5, and 6.

114. If any one is deſirous of knowing the equations for the coſines of the ſextuple, ſeptuple, octuple, and noncuple arcs, the equations for the ſines of which have been already found in *Corol.* 1, he will by computing them by either of the foregoing methods, but moſt eaſily by *Iſaac Newton*'s method contained in this *Corollary*, find them to be as follows; to wit, when the multiple arc is either leſs than one quadrant, or greater than three quadrants, we ſhall have

For the ſextuple arc, $32x^6 - 48r^2x^4 + 18r^4x^2 - r^6 = r^5q$, For the ſeptuple arc, $64x^7 - 112r^2x^5 + 56r^4x^3 - 7r^6x = r^6q$,

For the octuple arc, $128x^8 - 256r^2x^6 + 160r^4x^4 - 32r^6x^2 + r^8 = r^7q$,

And for the noncuple arc, $256x^9 - 576r^2x^7 + 432r^4x^5 - 120r^6x^3 + 9r^8x = r^8q$.

And conſequently, when the multiple arc is greater than one quadrant, but leſs than three quadrants, we ſhall have

For the ſextuple arc, $r^6 - 18r^4x^2 + 48r^2x^4 - 32x^6 = r^5q$,

For the ſeptuple arc, $7r^6x - 56r^4x^3 + 112r^2x^5 - 64x^7 = r^6q$,

For the octuple arc, $256r^2x^6 + 32r^6x^2 - 128x^8 - 160r^4x^4 - r^8 = r^7q$,

or $-128x^8 + 256r^2x^6 - 160r^4x^4 + 32r^6x^2 - r^8 = r^7q$,

And for the noncuple arc, $576r^2x^7 + 120r^6x^3 - 256x^9 - 432r^4x^5 - 9r^8x = r^8q$,

or $-256x^9 + 576r^2x^7 - 432r^4x^5 + 120r^6x^3 - 9r^8x = r^8q$.

115. *Corol.* 8. Sir *Iſaac Newton* in the forementioned *Problem* of his Algebra, does not actually compute the coſines of the double, triple, or any following multiple of the given angle FAG, but only ſhews how they may be computed by the method deſcribed in the laſt *Corollary*: Inſtead of the coſines he has there choſe rather to compute the chords of the complements of the multiples of that angle (which he ſuppoſes to be ſituated in the circumference, not at the center, of a circle) to a right angle, which is performed in the following manner.

If AB (*Fig.* 56.) be made the diameter of a circle inſtead of it's radius, BK and AK will not be the ſine and coſine of the angle A, or the ſines of the angles KAB, ABK, in that circle; but they will be the chords of the arches that ſubtend thoſe angles, which will now be ſituated in the circumference of the circle, becauſe, the angle AKB being a right angle, the circle will paſs through the point K. In like manner BL, CM, DN, EO, &c. will be the chords that ſubtend the angles BCM, CDM, DEN, EFO, &c. ſituated in the circumferences of their reſpective circles; that is, BL, CM, DN, EO, &c. will be the chord that ſubtend the complements of 2A, 3A, 4A, 5A, &c. ſituated in the circumferences of their reſpective circles, to a right angle.

Now let AK $= x$, as before, and put AB $= 2r$. The proceſſes will be as follows.

Firſt, AB : AK :: AC : AL; or $2r : x :: 2x :$ AL; therefore AL is $= \frac{xx}{r}$, and BL, the chord that ſubtends the complement of 2A, ſituated in the circumference of a circle whoſe diameter is BC, to a right angle, is = AL—AB $= \frac{xx}{r} - 2r$, or $\frac{xx - 2rr}{r}$.

Secondly, AB : AK :: AD : AM, or (becauſe AD is = AL + LD = AL + LB = 2AL —AB) AB : AK :: 2AL — AB : AM; that is, $2r : x :: \frac{2xx}{r} - 2r :$ AM; therefore AM is $= \frac{x^3}{rr} - x$, and conſequently CM, or the chord that ſubtends the complement of 3A, ſituated in the circumference of the circle, to a right angle, is (= AM — AC = AM — 2AK) =

$\frac{x^3}{rr} - 3x$, or $\frac{x^3 - 3rrx}{rr}$.

Thirdly, AB : AK :: AE (= 2AM — AC) : AN; or $2r : x :: \frac{2x^3}{rr} - 4x$: AN. Therefore AN is $= \frac{x^4}{r^3} - \frac{2xx}{r}$, and conſequently DN, the chord that ſubtends the complement of 4A, ſituated in the circumference of the circle, to a right angle, is (= AN — AD $= \frac{x^4}{r^3} - \frac{2xx}{r} - \frac{2xx}{r} + 2r) = \frac{x^4}{r^3} - \frac{4xx}{r} + 2r$, or $\frac{x^4 - 4r^2x^2 + 2r^4}{r^3}$.

Fourthly, AB : AK :: AF (= 2AN—AD) : AO; that is, $2r : x :: \left(\frac{2x^4}{r^3} - \frac{4xx}{r} - \frac{2xx}{r} + 2r =\right) \frac{2x^4}{r^3} - \frac{6xx}{r} + 2r$: AO. Therefore AO is $= \frac{x^5}{r^4} - \frac{3x^3}{rr} + x$, and conſequently EO, the chord that ſubtends the complement of 5A, ſituated in the circumference of the circle, to a right angle, is = AO — AE $= \frac{x^5}{r^4} - \frac{3x^3}{rr} + x - \frac{2x^3}{rr} + 4x = \frac{x^5}{r^4} - \frac{5x^3}{rr} + 5x$, or $\frac{x^5 - 5rrx^3 + 5r^4x}{r^4}$.

116. Therefore if q be put ſucceſſively for the chord that ſubtends the complement of the double, triple, quadruple, and quintuple, of the given angle A, or FAG, to a right angle, the equations expreſſing the relations of thoſe chords

to the chord AK, or x, that ſubtends the complement ABK of the ſimple angle FAG to a right angle, will be as follows.

For the double angle, $xx - 2rr = rq$,
For the triple angle, $x^3 - 3rrx = rrq$,
For the quadruple angle, $x^3 - 4r^2x^2 + 2r^4 = r^3q$,
And for the quintuple angle, $x^5 - 5r^2x^2 + 5r^4x = r^4q$.

117. If by the foregoing method we compute the chords of the complements of the ſextuple, ſeptuple, octuple, and noncuple, of the given angle A, to a right angle, we ſhall find the equations expreſſing their relations to the chord x of the complement of the ſimple arc A to a right angle, to be as follows.

For the ſextuple angle, $x^6 - 6r^2x^4 + 9r^4x^2 - 2r^6 = r^5q$,
For the ſeptuple angle, $x^7 - 7r^2x^5 + 14r^4x^3 - 7r^6x = r^6q$,
For the octuple angle, $x^8 - 8r^2x^6 + 20r^4x^4 - 16r^6x^2 + 2r^8 = r^7q$,
And for the noncuple angle, $x^9 - 9r^2x^7 + 27r^4x^5 - 30r^6x^3 + 9r^8x = r^8q$.

118. If the diameter AB be called d inſtead of $2r$, or if the chords q and x be referred to the diameter AB inſtead of the radius $\frac{AB}{2}$ or r, 'tis evident the equations expreſſing the relations of q and x to each other will be exactly like thoſe obtained in *Corol.* 7, for the coſines of multiple arcs, only that d muſt every where be ſubſtituted inſtead of r: For the proceſſes for finding the values of the chords q will be in every reſpect the ſame with thoſe in *Corol.* 7 for finding the coſines of the multiple arcs, except-

ing only that the letter d will every where be made uſe of inſtead of the letter r. Therefore if A be any angle ſituated in the circumference of a circle whoſe diameter is d, and x be the chord that ſubtends the complement of that angle, ſituated likewiſe in the circumference of the ſame circle, to a right angle, and q be put ſucceſſively for the chord that ſubtends the complement of the double, triple, quadruple, and other following multiples, of that angle, ſituated likewiſe in the circumference of the ſame circle, to a right angle, the relations between x and q will be expreſt by the following equations, to wit.

For the double angle, $2xx - dd = dq$,
For the triple angle. $4x^3 - 3ddx = ddq$,
For the quadruple angle, $8x^4 - 8d^2x^2 + d^4 = d^3q$,
For the quintuple angle, $16x^5 - 20d^2x^3 + 5d^4x = d^4q$,
For the ſextuple angle, $32x^6 - 48d^2x^4 + 18d^4x^2 - d^6 = d^5q$,
For the ſeptuple angle; $64x^7 - 112d^2x^5 + 56d^4x^3 - 7d^6x = d^6q$,
For the octuple angle, $128x^8 - 256d^2x^6 + 160d^4x^4 - 32d^6x^2 + d^8 = d^7q$,
And for the noncuple angle, $256x^9 - 576d^2x^7 + 432d^4x^5 - 120d^6x^3 + 9d^8x = d^8q$.

119. 'Tis evident theſe latter equations in which the diameter AB is put $= d$ are much leſs ſimple than thoſe obtained in the laſt article by putting AB $= 2r$, or $\frac{AB}{2} = r$; and this, I imagine, is the reaſon why Sir *Iſaac Newton* has thought fit to expreſs the relations of q and x to each other by the former equations in which

AB is $= 2r$, rather than by the latter equations in which AB is $= d$, or to refer the chords q and x to the radius r, or $\frac{AB}{2}$, rather than to the diameter d, or AB, to which latter line it is moſt uſual, and would otherwiſe perhaps be more convenient, to compare the chords of circular arches.

120. The equations obtained in the two preceding articles may, if we pleaſe, be eaſily referred to angles ſituated at the center, and not in the circumference, of a circle, in the following manner. If AB be, as before, the diameter of a circle, 'tis evident the lines BK and AK will be the chords of arches that ſubtend at the center of the circle two angles that are reſpectively double of KAB and ABK, or which are equal to 2A, and (twice the complement of A to a right angle, or) the complement of 2A to two right angles. In like manner the chords BL, CM, DN, EO, *&c.* will ſubtend at the centers of the ſeveral circles whoſe diameters are the lines BC, CD, DE, EF, *&c.* (which are all equal to AB) angles (that are reſpectively double of the angles BCM, CDM, DEN, EFO, *&c.* or of the complements of 2A, 3A, 4A, 5A, *&c.* to a right angle, or) that are reſpectively equal to the complents of 4A, 6A, 8A, 10A, *&c.* to two right angles. Therefore if B be $= 2A$, and conſequently $2B = 4A$, $3B = 6A$, $4B = 8A$, $5B = 10A$, *&c.* the chords AK, BL, CM, DN, EO, *&c.* will ſubtend at the centers of their ſeveral circles the complements of B, 2B, 3B, 4B, 5B, *&c.* to two right angles. Therefore if x be put for the chord that ſubtends at the center of a circle whoſe radius is r, and diameter d, the comple-

ment of any propoſed angle B to two right angles, and q be put ſucceſſively for the chord that ſubtends at the center of the ſame circle the complements of the double, triple, quadruple, quintuple, and other following multiples, of the angle B to two right angles, the equations expreſſing the relations of the chords q and x will, if we refer them to the radius, or make the diameter equal to $2r$, be as follows, to wit,

For the double angle, $xx - 2rr = rq$,

For the triple angle, $x^3 - 3rrx = rrq$,

&c. as in *Art.* 116, and 117.

And if we refer the chords q and x to the diameter, or make the diameter equal to d, the equations expreſſing therelations of theſe chords will be as follows, to wit,

For the double angle, $2xx - dd = dq$,

For the triple angle, $4x^3 - 3ddx = ddq$,

&c. as in *Art.* 118.

121. But theſe different ſuppoſitions of angles at the center, and angles in the circumference, of a circle, may be avoided by referring the chords to the arches they ſubtend without making any mention of the angles, which is in general a more convenient method of conſidering them than with relation to the angles. Now if this be done, 'twill be evident that if x be the chord of the ſupplement of any propoſed arc to a ſemicircle in a circle whoſe radius is r and diameter d, and q be put ſucceſſively for the chord of the ſupplement of the double, triple, quadruple, quintuple, and other following multiples of the propoſed arc, to a ſemicircle, the equations expreſſing the relations of q and x to each other will, if we refer them to the radius of the circle, or put the diameter equal to $2r$, be as follows; to wit,

For the double arc, $xx - 2rr = rq$;

For the triple arc, $x^3 - 3rrx = rrq$, &c. as in *Art.* 116, and 117.

And if we refer the chords q and x to the diameter of the circle, or put the diameter $= d$, the equations expreſſing the relations of q and x to each other will be as follows; to wit,

For the double arc, $2xx - dd = dq$,

For the triple arc, $4x^3 - 3ddx = ddq$,

&c. as in *Art.* 118.

122. Theſe equations, 'tis evident, are true as long as the chord q of the complement of the multiple arc to a ſemicircle has any magnitude, or as long as the multiple arc is of any magnitude leſs than a ſemicircle; for ſo long the diameters DE, EF, FG, &c. will be greater than the chords EN, FO, PG, &c. which are the perpendicular, or leaſt, diſtances of the lines AF, AG, that contain the angle FAG; and therefore ſo long may thoſe diameters be inſcribed in that angle, and the conſtruction of *Fig.* 56, and the reaſonings found upon it will continue the ſame. Afterwards when the multiple arc is greater than a ſemicircle, but leſs than a whole circle, thoſe terms of the equations which before were the leſſer and therefore were ſubtracted from the others, will become the greater, and conſequently the others muſt now be ſubtracted from them; that is, in other words, the ſigns of all the terms on the firſt, or left-hand, ſides of the equations muſt every where be changed. In this caſe therefore, if the diameter be put $= 2r$, we ſhall have,

For the double arc, $2rr - xx = rq$,

For the triple arc, $3rrx - x^3 = rrq$,

For the quadruple arc, $4r^2x^2 - x^4 - 2r^4 = r^3q$,

For the quintuple arc, $5r^2x^3 - x^5 - 5r^4x = r^4q$,

and ſo of the following equations:

And, if the diameter be put $= d$, we ſhall have

For the double arc, $dd - 2xx = dq$,

For the triple arc, $3ddx - 4x^3 = ddq$,

For the quadruple arc, $8d^2x^2 - 8x^4 - d^4 = d^3q$,

For the quintuple arc, $20d^2x^3 - 16x^5 - 5d^4x = d^4q$,

and ſo of the following equations.

The demonſtration of this regular change in theſe equations will be given in the latter part of the next *Corollary*, *Art.* 133, and *Seq.*

123. *Corol.* 9. From the chords of the complements of the ſeveral multiple arcs to a ſemicircle we may derive the chords of the ſeveral multiple arcs themſelves in the following manner. Let y be the chord of the ſimple arc, and let p ſtand ſucceſſively for the chord of the double, triple, quadruple, quintuple, and other following multiple arcs: Then 'tis evident that dd will be equal to $pp + qq$, and alſo to $yy + xx$, and conſequently qq will be $= dd - pp$, xx to $dd - yy$, x^4 to $\overline{dd - yy}|^2$ or $d^4 - 2d^2y^2 + y^4$, x^6 to $\overline{dd - yy}|^3$ or $d^6 - 3d^4y^2 + 3d^2y^4 - y^6$, x^8 to $\overline{dd - yy}|^4$ or $d^8 - 4d^6y^2 + 6d^4y^4 - 4d^2y^6 + y^8$, and x^{10} to $\overline{dd - yy}|^5$ or $d^{10} - 5d^8y^2 + 10d^6y^4 - 10d^6y^4 + 5d^2y^8 - y^{10}$, &c. Therefore in the firſt place, ſince $2xx - dd$ is $= dq$, and conſequently $4x^4 - 4d^2x^2 + d^4 = d^2q^2$, it will follow by ſubſtituting the values of x and q, that $4 \times \overline{d^4 - 2d^2y^2 + y^4} - 4d^2 \times \overline{dd - yy} + d^4$, or $4d^4 - 8d^2y^2 + 4y^4 - 4d^4 + 4d^2y^2 + d^4$, or $4y^4 - 4d^2y^2 + d^4$, is $= dd \times \overline{dd - pp}$, or $d^4 - d^2p^2$, and conſequently $4d^2y^2 - 4y^4 = d^2p^2$.

Secondly, Since $4x^3 - 3ddx$ is $= ddq$, we ſhall have $16x^6 - 24d^2x^4 + 9d^4x^2 = d^4q^2$, that is, $16 \times \overline{d^6 - 3d^4y^2 + 3d^2y^4 - y^6} - 24d^2 \times \overline{d^4 - 2d^2y^2 + y^4} + 9d^4 \times \overline{d^2 - y^2} = d^4 \times \overline{dd - pp}$, or $16d^6 - 48d^4x^2 + 48d^2y^4 - 16y^6 - 24d^6 + 48d^4y^2 - 24d^2y^4 + 9d^6 - 9d^4y^2 = d^6 - d^4p^2$, or $d^6 - 9d^4y^2 + 24d^2y^4 - 16y^6 = d^6 - d^4p^2$, and $9d^4y^2 - 24d^2y^4 + 16y^6 = d^4p^2$, and consequently $3ddy - 4y^3 = ddp$.

Thirdly, Since $8x^4 - 8d^2x^2 + d^4$ is $= d^3q$, we ſhall have $64x^8 - 128d^2x^6 + 64d^4x^4 + 16d^4x^4 - 16d^6x^2 + d^8$, or $64x^8 - 128d^2x^6 + 80d^4x^4 - 16d^6x^2 + d^8, = d^6q^2$, that is, $64 \times \overline{d^8 - 4d^6y^2 + 6d^4y^4 - 4d^2y^6 + y^8} - 128d^2 \times \overline{d^6 - 3d^4y^2 + 3d^2y^4 - y^6} + 80d^4 \times \overline{d^4 - 2d^2y^2 + y^4} - 16d^6 \times \overline{dd - yy} + d^8 = d^6 \times \overline{dd - pp}$, or $64d^8 - 256d^6y^2 + 38 d^4y^4 - 256d^2y^6 + 64y^8 - 128d^8 + 384d^6y^2 - 384d^4y^4 + 128d^2y^6 + 80d^8 - 160d^6y^2 + 80d^4y^4 - 16d^8 + 16d^6y^2 + d^8 = d^8 - d^6p^2$, or $d^8 - 16d^6y^2 + 80d^4y^4 - 128d^2y^6 + 64y^8 = d^8 - d^6p^2$, and conſequently $16d^6y^2 - 80d^4y^4 + 128d^2y^6 - 64y^8 = d^6p^2$.

Fourthly, Since $16x^5 - 20d^2x^3 + 5d^4x$ is $= d^4q$, we ſhall have $256x^{10} - 640d^2x^8 + 400d^4x^6 + 160d^4x^6 - 200d^6x^4 + 25d^8x^2 = d^8q^2$, or $256x^{10} - 640d^2x^8 + 560d^4x^6 - 200d^6x^4 + 25d^8x^2 = d^8q^2$, that is, $256 \times \overline{d^{10} - 5d^8y^2 + 10d^6y^4 - 10d^4y^6 + 5d^2y^8 - y^{10}} - 640d^2 \times \overline{d^8 - 4d^6y^2 + 6d^4y^4 - 4d^2y^6 + y^8} + 560d^4 \times \overline{d^6 - 3d^4y^2 + 3d^2y^4 - y^6} - 200d^6 \times \overline{d^4 - 2d^2y^2 + y^4} + 25d^8 \times \overline{dd - yy} = d^8 \times \overline{dd - pp}$, or $256d^{10} - 1280d^8y^2 + 2560d^4y^4 - 2560d^4y^6 + 1280d^2y^8 - 256y^{10} - 640d^{10} + 2560d^8y^2 - 3840d^6y^4 + 2560d^4y^6 - 640d^2y^8 + 560d^{10} - 1680d^8y^2 + 1680d^6y^4 - 560d^4y^6 - 200d^{10} + 400d^2y^8 - 200d^6y^4 + 25d^{10}$,

$- 25d^8y^2 = d^{10} - d^8p^2$, or $d^{10} - 25d^8y^2 + 200d^6y^4 - 560d^4y^6 + 640d^2y^8 - 256y^{10} = d^{10} - d^8p^2$, whence $25d^8y^2 - 200d^6y^4 + 560d^4y^6 - 640d^2y^8 + 256y^{10} = d^8p^2$, and conſequently $5d^4y - 20d^2y^3 + 16y^5 = d^4p$.

124. Therefore if y be the chord of any arc in a circle whoſe diameter is d, and p be put ſucceſſively for the chord of the double, triple, quadruple, and quintuple of that arc, the relations between the chords y and p will be expreſſed by the following equations; to wit,

For the double arc we ſhall have $4d^2y^2 - 4y^4 = d^2p^2$,

For the triple arc, $3ddy - 4y^3 = ddp$,

For the quadruple arc, $16d^6y^2 - 80d^4y^4 + 128d^2y^6 - 64y^8 = d^6p^2$,

And for the quintuple arc, $5d^4y - 20d^2y^3 + 16y^5 = d^4p$.

125. And in the ſame manner that theſe equations have been computed, may the equations for the chords of all the following multiple arcs be derived from thoſe found in the laſt *Corollary* for the chords of the complements of thoſe multiples to a ſemicircle. But it muſt be obſerved that the equations ſo obtained cannot by this method be concluded to extend further than to thoſe caſes in which the multiple arc is not greater than a ſemicircle (although in truth they extend likewiſe to the other caſe in which the multiple arc is greater than a ſemicircle, but leſs than a whole circle, as will preſently be ſhewn), becauſe the method by which we obtained, by means of *Fig.* 56, the equations for the chords of the complements extends only to thoſe caſes in which the multiple arc does not exceed a ſemicircle.

126. But the equations for the chords of multiple arcs may be derived immediately from the equations for their ſines in the following manner. If s be the ſine of any circular arc called A, and x be put for the ſine of the double, triple, quadruple, or any other multiple of the arc A called m A, in a circle whoſe radius is r and diameter d, and y be put for the chord of 2A, and p for the chord of $2m$A, 'tis evident that d will be double of r, y of s, and p of x, and conſequently d, y, and p will be related to each other in the ſame manner as r, s, and x, or the relations of d, y, and p to each other will be expreſſed by an equation involving the ſame combinations of the lines d, y, and p one with another as the lines r, s, and x undergo reſpectively in the equation that expreſſes their mutual relations to each other. Therefore if we put $B = 2A$, and conſequently $mB = 2mA$, the relations of the chord y of the ſimple arc B to the chords p of the ſeveral ſucceſſive multiples of the arc B denoted by mB, will be as follows; to wit, we we ſhall have

For the double arc, $4d^2y^2 - 4y^4 = d^2p^2$,

For the triple arc, $3ddy - 4y^3 = ddp$,

For the quadruple arc, $16d^5y^2 - 80d^4y^4 + 128d^2y^6 - 64y^8 = d^6p^2$,

For the quintuple arc, $5d^4y - 20d^2y^3 + 16y^5 = d^4p$,

For the ſextuple arc, $36d^{10}y^2 - 420d^8y^4 + 1792d^6y^6 - 3456d^4y^8 + 3072d^2y^{10} - 1024y^{12} = d^{10}p^2$,

For the ſeptuple arc, $7d^6y - 56d^4y^3 + 112d^2y^5 - 64y^7 = d^6p$,

For the octuple arc, $64d^{14}y^2 - 1344d^{12}y^4 + 10752d^{10}y^6 - 42240d^8y^8 + 90112d^6y^{10}$

$- 106$

$-108496d^4y^{12}+65536d^2y^{14}-16384y^{16}$
$=d^{14}p^2$,

And for the noncuple arc, $9d^8y-120d^6y^3+$
$432d^4y^5-576d^2y^7+256y^9=d^8p$.

127. If we refer the chords to the radius, or ſubſtitute $2r$ inſtead of d in all the terms of the foregoing equations, they will be thereby rendered ſomewhat ſimpler, and will be as follows; to wit, we ſhall have

For the double arc, $4r^2y^2-y^4=r^2p^2$,
For the triple arc, $3r^2y-y^3=rrp$,
For the quadruple arc, $16r^6y^2-20r^4y^4$
$+8r^2y^6-y^8=r^6p^2$,
For the quintuple arc, $5r^4y-5r^2y^3+y^5=r^4p$,
For the ſextuple arc, $36r^{10}y^2-105r^8y^4+$
$112r^6y^6-54r^4y^8+12r^2y^{10}-y^{12}=$
$r^{10}p^2$,
For the ſeptuple arc, $7r^6y-14r^4y^3+8r^2y^5$
$-y^7=r^6p$,
For the octuple arc, $64r^{14}y^2-338r^{12}y^4+$
$672r^{10}y^6-660r^8y^8+352r^6y^{10}-104$
$r^4y^{12}+16r^2y^{14}-y^{16}=r^{14}p^2$,
And for the noncuple arc, $9r^8y-30r^6y^3+$
$27r^4y^5-9r^2y^7+y^9=r^8p$.

128. And here it muſt be obſerved that, becauſe the equations for the ſines s and x from which the foregoing equations for the chords y and p are derived are true (by *Corol.* 4.) as long as the multiple arc mA is of any magnitude leſs than a ſemicircle, that is, as long as the arc mB is of any magnitude leſs than the whole circle, and alſo the quantities d, y, and p continue to be reſpectively double of r, s, and x as long as the arc mB is of any magnitude leſs than the whole circle, it follows that the foregoing equations for the chords of the multiple arcs (as well when thoſe chords are referred to the radius as

when they are referred to the diameter, or as well when the diameter is put $= 2r$ as when it is put $= d$; for 'tis evident this difference of notation cannot make any change in the ſigns of the terms of theſe equations:) muſt be true as long as mB, or the multiple arc, is of any magnitude leſs than the whole circle, agreeably to the aſſertion in *Art.* 125.

129. Hitherto we have determined the equations of the chords of multiple arcs by deriving them either from the equations of the chords of their complements to a ſemicircle, or from the the equations of the ſines of multiple arcs. But if we would determine them in a more immediate manner without any regard to the ſines, or the chords of the complements to a ſemicircle, we may obtain them by the help of *Prop.* 4, and 10, in the following manner.

In *Fig.* 9, let the diameter AM of the circle ABC, whoſe center is O, be put $= d$, the chord of the ſimple arc AB $= y$, and the chords AC, AD, AE, AF, *&c.* of the double, triple, quadruple, quintuple, and other following multiple arcs be ſucceſſively repreſented by p. Draw the chord BM, and from the center O draw OQ perpendicular to the chord AB, and meeting it in point Q. Then 'tis evident the triangles ABM, AQO will be ſimilar, and conſequently AM will be to BM as AO to OQ, and AM : 2BM :: AO : 2OQ. But, by *Prop.* 4. *Corol.* 1, AO : 2OQ :: AB : AC, becauſe OQ is the coſine of half the arc AB; therefore AM : 2BM :: AB : AC. Therefore

In the firſt place we have $d : 2\sqrt{dd - yy} :: y : AC$; conſequently AC, or the chord of the

double

double arc AC, is $= \frac{2y \times \sqrt{dd-yy}}{d}$.

Secondly, By *Prop.* 10, we have AB : AC :: AC : AB + AD, and conſequently AM : 2BM :: AC : AB + AD, that is $d : 2\sqrt{dd-yy} :: \frac{2y \times \sqrt{dd-yy}}{d} : y + \text{AD}$; therefore $y + \text{AD}$ is $= \frac{4d^2y - 4y^3}{dd}$, and AD or the chord of the triple arc AD, is $= \frac{4d^2y - 4y^3}{dd} - y = \frac{4d^2y - 4y^3}{dd} - \frac{ddy}{dd} = \frac{3d^2y - 4y^3}{dd}$.

Thirdly, By *Prop.* 10, we have AB : AC :: AD : AC + AE, and conſequently AM : 2BM :: AD : AC + AE; that is, $d : 2\sqrt{dd-yy} :: \frac{3d^2y - 4y^3}{dd} : \frac{2y \times \sqrt{dd-yy}}{d} + \text{AE}$; therefore AE + $\frac{2y\sqrt{dd-yy}}{d}$ is $= \frac{6d^2y - 8y^3}{d^3} \times \sqrt{dd-yy}$, and AE, or the chord of the quadruple arc AE, is $(= \frac{6d^2y - 8y^3}{d^3} \times \sqrt{dd-yy} - \frac{2y \times \sqrt{dd-yy}}{d} = \frac{6d^2y - 8y^3}{d^3} \times \sqrt{dd-yy} - \frac{2d^2y}{d^3} \times \sqrt{dd-yy}) = \frac{4d^2y - 8y^3}{d^3} \times \sqrt{dd-yy}$.

Fourthly, By *Prop.* 10, we have AB : AC :: AE : AD + AF, and conſequently AM : 2BM :: AE : AD + AF, that is, $d : 2\sqrt{dd-yy} :: \frac{4d^2y - 8y^3}{d^3} \times \sqrt{dd-yy} : \frac{3d^2y - 4y^3}{dd} +$

AF; therefore $\frac{3d^2y - 4y^3}{dd}$ + AF is = $\frac{8d^2y - 16y^3}{d^4} \times \overline{dd - yy} = \frac{8d^4y - 24d^2y^3 + 16y^5}{d^4}$, and AF, or the chord of the quintuple arc AF, is $\left(= \frac{8d^4y - 24d^2y^3 + 16y^5}{d^4} \frac{-3d^2y + 4y^3}{dd} = \frac{8d^4y - 24d^2y^3 + 16y^5}{d^4} \frac{-3d^4y + 4d^2y^3}{d^4} \right) = \frac{5d^4y - 20d^2y^3 + 16y^5}{d^4}$.

Fifthly, By *Prop.* 10, we have AB : AC :: AF : AE + AG, and conſequently AM : 2BM :: AF : AE + AG; that is, $d : 2\overline{dd - yy} :: \frac{5d^4y - 20d^2y^3 + 16y^5}{d^4} : \frac{4d^2y - 8y^3}{d^3} \times \sqrt{dd - yy}$ + AG; therefore AG $\frac{+4d^2y - 8y^3}{d^3} \times \sqrt{dd - yy}$ is $= \frac{10d^4y - 40d^2y^3 + 32y^5}{d^5} \times \sqrt{dd - yy}$, and AG, or the chord of the ſextuple arc AG, is $\left(= \overline{\frac{10d^4y - 40d^2y^3 + 32y^5}{d^5}}\Big| \times \sqrt{dd - yy} \; \overline{\frac{-4d^2y + 8y^3}{d^3}}\Big| \times \sqrt{dd - yy} = \overline{\frac{10d^4y - 40d^2y^3 + 32y^5}{d^5}}\Big| \times \sqrt{dd - yy} \; \overline{\frac{-4dy^4 + 8d^2y^3}{d^5}}\Big| \times \sqrt{dd - yy} \right) = \overline{\frac{6d^4y - 32d^2y^3 + 32y^5}{d^5}}\Big| \times \sqrt{dd - yy}$.

Sixthly, By *Prop.* 10, we have AB : AC :: AG : AF + AP, and conſequently AM : 2BM :: AG : AF + AP; that is, $d : 2\sqrt{dd-yy} :: \frac{6d^4y - 32d^2y^3 + 32y^5}{d^5} \times \sqrt{dd-yy} : \frac{5d^4y - 20d^2y^3 + 16y^5}{d^4} + \text{AP}$; therefore AP $\frac{+ 5d^4y - 20d^2y^3 + 16y^5}{d^4}$ is $= \frac{12d^4y - 64d^2y^3 + 64y^5}{d^6} \times \overline{dd - yy} = \frac{12d^6y - 76d^4y^3 + 128d^2y^5 - 64y^7}{d^6}$, and AP, or the chord of the ſeptuple arc AP, is $= \frac{12d^6y - 76d^4y^3 + 128d^2y^5 - 64y^7}{d^6}$

$$\frac{- 5d^4y + 20d^2y^3 - 16y^5}{d^4}$$

$$= \frac{12d^6y - 76d^4y^3 + 128d^2y^5 - 64y^7}{d^6}$$

$$\frac{- 5d^6y + 20d^4y^3 - 16d^2y^5}{d^6}$$

$$= \frac{7d^6y - 56d^4y^3 + 112d^2y^5 - 64y^7}{d^6}$$

Seventhly, By *Prop.* 10, we have AB : AC :: AP : AG + AR, and conſequently AM : 2BM :: AP : AG + AR; that is, $d : 2\sqrt{dd \quad yy} :: \frac{7d^6y - 56d^4y^3 + 112d^2y^5 - 64y^7}{d^6} : \frac{6d^4y - 32d^2y^3 + 32y^5}{d^5} \times \sqrt{dd - yy} + \text{AR}$; therefore AR $+ \frac{6d^4y - 32d^2y^3 + 32y^5}{d^5} \times$

$\sqrt{dd-yy}$ is $= \dfrac{14d^6y - 112d^4y^3 + 224d^2y^5 - 128y^7}{d^7} \times \sqrt{dd-yy}$,

and conſequently AR, or the chord of the octuple arc AR, is

$$\Big(= \frac{14d^6y - 112d^4y^3 + 224d^2y^5 - 128y^7}{d^7} \times \sqrt{dd-yy} - \frac{6d^4y + 32d^2y^3 - 32y^5}{d^5} \times \sqrt{dd-yy} = \frac{14d^6y - 112d^4y^3 + 224d^2y^5 - 128y^7}{d^7} \times \sqrt{dd-yy} - \frac{6d^6y + 32d^4y^3 - 32d^2y^5}{d^7} \times \sqrt{dd-yy}\Big) = \frac{8d^6y - 80d^4y^3 + 192d^2y^5 - 128y^7}{d^7} \times \sqrt{dd-yy}.$$

Eighthly, By *Prop.* 10, we have AB : AC :: AR : AP + AT, and conſequently AM : 2BM :: AR : AP + AT; that is, $d : 2\sqrt{dd-yy} :: \dfrac{8d^6y - 80d^4y^3 + 192d^2y^5 - 128y^7}{d^7} \times \sqrt{dd-yy} : \dfrac{7d^6y - 56d^4y^3 + 112d^2y^5 - 64y^7}{d^6} + \text{AT}$; therefore $\text{AT} + \dfrac{7d^6y - 56d^4y^3 + 112d^2y^5 - 64y^7}{d^6}$ is $= \dfrac{16d^6y - 160d^4y^3 + 384d^2y^5 - 256y^7}{d^8} \times \overline{dd-yy} = \dfrac{16d^8y - 176d^6y^3 + 544d^4y^5 - 640d^2y^7 + 256y^9}{d^8}$, and conſequently AT, or the chord of the nonuple cuple arc AT, is (=

$$\frac{16d^8y - 176d^6y^3 + 544d^4y^5 - 640d^2y^7 + 256y^9}{d^8}$$

$$\frac{-7d^6y + 56d^4y^3 - 112d^2y^5 + 64y^7}{d^6} =$$

$$\frac{16d^8y \quad 176d^6y^3 + 544d^4y^5 - 640d^2y^7 + 256y^9}{d^8}$$

$$\frac{-7d^8y + 56d^6y^3 - 112d^4y^5 + 64d^2y^7}{d^8} =$$

$$\frac{9d^8y - 120d^6y^3 + 432d^4y^5 - 576d^2y^7 + 256y^9}{d^8}.$$

In the ſame manner may be found the chords of the arcs AZ and AX, and of all the following multiples of the arc AB.

130. Hence 'tis evident that the equations for the chords of the multiple arcs AC, AD, AE, *&c.* as far as the noncuple arc AT, are as follows, to wit,

For the double arc, $2y \times \sqrt{dd - yy} = dp$,

or, $4d^2y^2 - 4y^4 = d^2p^2$,

For the triple arc, $3d^2y - 4y^3 = d^2p$,

For the quadruple arc, $\overline{3d^2y - 4y^3} \times \sqrt{dd - yy} = d^3p$,

or, $16d^6y^2 - 80d^4y^4 + 128d^2y^6 - 64y^8 = d^6p^2$

For the quintuple arc, $5d^4y - 20d^2y^3 + 16y^5 = d^4p$,

For the ſextuple arc, $\overline{6d^4y - 32d^2y^3 + 32y^5} \times \sqrt{dd - yy} = d^5p$,

or, $36d^{10}y^2 - 420d^8y^4 + 1792d^6y^6 - 3456d^4y^8 + 3072d^2y^{10} - 1024y^{12} = d^{10}p^2$,

For the ſeptuple arc, $7d^6y - 56d^4y^3 + 112d^2y^5 - 64y^7 = d^6p$, For the octuple arc,

$$\overline{8d^6y - 80d^4y^3 + 192d^2y^5 - 128y^7} \times \sqrt{dd - yy} = d^7p,$$

or, $64d^{14}y^2 - 1344d^{12}y^4 + 10752d^{10}y^6 - 42240d^8y^8 + 90112d^6y^{10} - 106496d^4y^{12} + 65536d^2y^{14} - 16384y^{16} = d^{14}p^2$,

And for the noncuple arc, $9d^8y - 120d^6y^3 + 432d^4y^5 - 576d^2y^7 + 256y^9 = d^8p$; all which equations are the ſame with thoſe found by other methods in *Art.* 124, and 126.

131. And here we muſt obſerve that becauſe the 4th and 10th *Propoſitions* and the 1ſt *Corollary* to the 4th *Propoſition*, by the help of which the foregoing equations have been juſt now obtained, are true as long as the multiple arc is of any magnitude whatſoever leſs than the whole circle, it follows that the foregoing equations are true as well when the multiple arc is greater than a ſemicircle, but leſs than a whole circle, as when it is leſs than a ſemicircle, or as long as the multiple arc is of any magnitude whatſoever leſs than the whole circle, agreeably to what was proved in *Art.* 128, and aſſerted in *Art.* 125.

132. From this laſt method of obtaining of the equations of the chords of multiple arcs, which is certainly the moſt genuine and natural way of obtaining them, it is eaſy to perceive by what means it comes to paſs that the equations for the chords of the even multiples of a given arc riſe to a greater number of dimenſions than thoſe for the chords of it's odd multiples. For 'tis evident that, as the chords of all the multiple arcs are obtained by making uſe of a proportion in which d and $2\sqrt{dd - yy}$ are the firſt and ſecond terms, and the chord of the next leſſer

multiple arc is the third term, there will always be in the computing of every new chord a multiplication of the preceding chord by the irrational quantity $\sqrt{dd-yy}$; and conſequently the value of the chord of the double arc will be irrational; that of the chord of the triple arc, on account of the ſecond multiplication by $\sqrt{dd-yy}$ which upon the whole produces a multiplication by the rational quantity $dd-yy$, will be rational; that of the chord of the quadruple arc, on account of the third multiplication by $\sqrt{dd-yy}$ which upon the whole produces a multiplication by the irrational quantity $dd-yy \times \sqrt{dd-yy}$, will be irrational; and for the ſame reaſon the values of the chords of the quintuple, ſextuple, ſeptuple, and all the following multiple arcs, will be alternately rational and irrational. Conſequently if we would take away the irrationality of the values of theſe chords of the even multiples, and expreſs their relations to the chord of the ſimple arc by equations conſiſting of a *finite* number of rational terms, we muſt ſquare the expreſſions of their values, and thereby raiſe the dimenſions of the ſeveral terms to double of what they were before, or to double the number of times the multiple arc contains the ſimple one.

133. Having thus found the equations for the chords of multiple arcs by two different methods both independent of the equations for the chords of their ſupplements to a ſemicircle, and having ſhewn that theſe equations undergo no change, but continue to be the ſame as long as the multiple arc is of any magnitude leſs than the whole circle, we may now derive the equations for the chords of their ſupplements to a ſe-

micircle from the equations for the chords of the multiple arcs themſelves, and demonſtrate what was aſſerted in art. 122, concerning the change of the ſigns of their terms when the multiple arc becomes greater than a ſemicircle, in the following manner.

Let d be put for the diameter of the circle, x for the chord of the ſupplement of the ſimple arc to a ſemicircle, and q for the chord of the ſupplement of the multiple arc to a ſemicircle; and it is evident, we ſhall have $dd = yy + xx$, and and likewiſe $dd = pp + qq$, and conſequently $pp = dd - qq$, and $yy = dd - xx$, $y^4 = \overline{dd - xx}|^2 = d^4 - 2d^2x^2 + x^4$, $y^6 = \overline{dd - xx}|^3 = d^6 - 3d^4x^2 + 3d^2x^4 - x^6$, $y^8 = \overline{dd - xx}|^4 = d^8 - 4d^6x^2 + 6d^4x^4 - 4d^2x^6 + x^6$, $y^{10} = \overline{dd - xx}|^5 = d^{10} - 5d^8x^2 + 10d^5x^4 - 10d^4x^6 + 5d^2x^8 - x^{10}$, &c. Therefore,

In the firſt place, ſince $4d^2y^2 - 4y^4$ is $= d^2p^2$, it will follow by ſubſtituting $dd - qq$ for pp, and $dd - xx$ for yy, that $4d^2 \times \overline{dd - xx} - 4 \times \overline{d^4 - 2d^2x^2 + x^4}$, or $4d^4 - 4d\; x^2 - 4d^4 + 8d^2xx - 4x^4$, or $4d^2x^2 - 4x^4$, is $= d^2 \times \overline{dd - qq}$, or $d^4 - d^2q^2$, and conſequently $d^4 - 4d^2x^2 + 4x^4$ is $= d^2q^2$. Therefore the ſquare root of the quantity $d^4 - 4d^2x^2 + 4x^4$ is $= dq$: But the ſquare root of $d^4 - 4d^2x^2 + 4x^4$ is either $2x^2 - d^2$, or $d^2 - 2x^2$, according as $2xx$ is greater or leſs than dd: Therefore either $2xx - dd$, or $dd - 2xx$ is $= dq$; according as $2xx$ is greater or leſs than dd. Now $2xx$ is greater than dd as long as the double arc is leſs than a ſemicircle, and leſs than dd when the double arc is greater than a ſemicircle, but leſs than the whole circle, as may be thus demonſtrated. Since the ſquare root of the quantity $d^4 - 4d^2x^2 + 4x^4$ is either $2xx - dd$, or $dd - 2xx$

and the quantity $d^4 - 4d^2x^2 + 4x^4$ is always equal to d^2q^2, and conſequently its ſquare root is $= dq$, it follows, that when dq is $= 0$, that is, when the double arc is equal to a ſemicircle, (for in that caſe q, or the chord of the ſupplement of the double arc to a ſemicircle, is $= 0$, and conſequently dq is 0), one of the two quantities $2xx$ dd, and $dd - 2xx$, will be $= 0$, and conſequently $2xx$ will be $= dd$. Therefore while the double arc is leſs than a ſemicircle, and conſequently the chord y of the ſimple arc is leſs, and the chord x of the ſupplement of the ſimple arc to a ſemicircle is greater, than when the double arc is equal to a ſemicircle, $2xx$ will be greater than dd; and when the double arc is greater than a ſemicircle, but leſs than the whole circle, and conſequently the chord y of the ſimple arc is greater, and the chord x of the ſupplement of the ſimple arc to a ſemicircle is leſs, than when the double arc is equal to a ſemicircle, $2xx$ will be leſs than dd. Therefore as long as the double arc is leſs than a ſemicircle, we ſhall have $2xx - dd = dq$, and when the double arc is greater than a ſemicircle, but leſs than the whole circle, we ſhall have $dd - 2xx = dq$.

Secondly, Since $3d^2y - 4y^3$ is $= d^2p$, and conſequently $9d^4y^2 - 24d^2y^4 + 16y^6$ is $= d^4p^2$, it will follow, by ſubſtituting $\overline{dd - xx}$ for yy, and $\overline{dd - qq}$ for pp, that $9d^4 \times \overline{dd - xx} - 24d^2 \times \overline{d^4 - 2d^2x^2 + x^4} + 16 \times \overline{d^6 - 3d^4x^2 + 3d^2x^4 - x^6}$, or $9d^6 - 9d^4x^2 - 24d^6 + 48d^4x^2 - 24d^2x^4 + 16d^6 - 48d^4x^2 + 48d^2x^4 - 16x^6$, or $d^6 - 9d^4x^2 + 24d^2x^4 - 16x^6$, is $= d^4 \times \overline{dd - qq}$, or $d^6 - d^4q^2$, and conſequently $9d^4x^2 - 24d^2x^4 + 16x^6 = d^4q^2$. Therefore the ſquare root of the quantity $9d^4x^2 - 24d^2x^4 + 16x^6$ is $=$ d^2q: But the ſquare root of $9d^4x^2 - 24d^2x^4 + 16x^6$ is either $4x^3 - 3d^2x$, or $3d^2x - 4x^3$, according as $4x^3$ is greater or leſs than $3d^2x$: Therefore either $4x^3 - 3d^2x$, or $3d^2x - 4x^3$ will be $= d^2q$, according as $4x^3$ is greater or leſs than $3d^2x$. Now $4x^3$ is greater than $3d^2x$ as long as the triple arc is leſs than a ſemicircle, and leſs than $3ddx$ when the triple arc is greater than a ſemicircle, but leſs than the whole circle, as may be thus demonſtrated. Since the ſquare root of the quantity $9d^4x^2 - 24d^2x^4 + 16x^6$ is either $4x^3 - 3d^2x$, or $3d^2x - 4x^3$, and the quantity $9d^4x^2 - 24d^2x^4 + 16x^6$ is always equal to d^4q^2, and conſequently its ſquare root is $= d^2q$, it follows, that when d^2q is $= 0$, that is, when the triple arc is equal to a ſemicircle, (for in that caſe q, or the chord of the ſupplement of the triple arc to a ſemicircle, is $= 0$, and conſequently d^2q is $= 0$) one of the two quantities $4x^3 - 3d^2x$ and $3d^2x - 4x^3$, will be $= 0$, and conſequently $4x^3$ will be $= 3d^2x$, and $4xx = 3dd$. Therefore while the triple arc is leſs than a ſemicircle, and conſequently the chord y of the ſimple arc is leſs, and the chord x of the ſupplement of the ſimple arc to a ſemicircle is greater, than when the triple arc is equal to a ſemicircle, $4xx$ will be greater than $3dd$, and $4x^3$ than $3ddx$; and when the triple arc is greater than a ſemicircle, but leſs than the whole circle, and conſequently the chord y of the ſimple arc is greater, and the chord x of the ſupplement of the ſimple arc to a ſemicircle is leſs, than when the triple arc is equal to a ſemicircle, $4xx$ will be leſs than $3dd$, and $4x^3$ than $3d^2x$. Therefore as long as the triple arc is leſs than a ſemicircle, we ſhall have

$4x^3 - 3d^2x = d^2q$, and when the triple arc is greater than a ſemicircle, but leſs than the whole circle, we ſhall have $3d^2x - 4x^3 = d^2q$.

Thirdly, Since $16d^6y^2 - 80d^4y^4 + 128d^2y^6 - 64y^8$ is $= d^6p^2$, it will follow by ſubſtituting $dd - xx$ for yy, and $dd - qq$ for pp, that $16d^6 \times \overline{dd - xx} - 80d^4 \times \overline{d^4 - 2d^2x^2 + x^4} + 128d^2 \times \overline{d^6 - 3d^4x^2 + 3d^2x^4 - x^6} - 64 \times \overline{d^8 - 4d^6x^2 + 6d^4x^4 - 4d^2x^6 + x^8}$, or $16d^8 - 16d^6x^2 \quad 80d^8 + 160d^6x^2 - 80d^4x^4 + 128d^8 - 384d^6x^2 + 384d^4x^4 - 128d^2x^6 - 64d^3 + 256d^6x^2 - 384d^4x^4 + 256d^2x^6 - 64x^8$, or $16d^6x^2 - 80d^4x^4 + 128d^2x^6 - 64x^8$, is $= d^6 \times \overline{dd - qq}$, or $d^8 - d^6q^2$, and conſequently that $d^8 - 16d^6x^2 + 80d^4x^4 - 128d^2x^6 + 64x^8$ is $= d^6q^2$. Therefore the ſquare root of the quantity $d^8 - 16d^6x^2 + 80d^4x^4 - 128d^2x^6 + 64x^8$ is $= d^3q$. But the ſquare root of $d^8 - 16d^6x^2 + 80d^4x^4 - 128d^2x^6 + 64x^8$ is either $8x^4 - 8d^2x^2 + d^4$, or $8d^2x^2 - 8x^4 - d^4$, according as $8x^4 + d^4$ is greater or leſs than $8d^2x^2$. Therefore either $8x^4 - 8d^2x^2 + d^4$, or $8d^2x^2 - 8x^4 - d^4$, will be $= d^3q$, according as $8x^4 + d^4$ is greater or leſs than $8d^2x^2$. Now $8d^4 + d^4$ is greater than $8d^2x^2$ as long as the quadruple arc is leſs than a ſemicircle, and leſs than $8d^2x^2$ when the quadruple arc is greater than a ſemicircle, but leſs than the whole circle; as may be thus demonſtrated. Since the ſquare root of the quantity $d^8 - 16d^5x^2 + 80d^4x^4 - 128d^2x^6 + 64x^8$ is either $8x^4 - 8d^2x^2 + d^4$, or $8d^2x^2 - 8x^4 - d^4$, and the quantity $d^3 - 16d^6x^2 + 80d^4x^4 - 128d^2x^6 + 64x^8$ is always equal to d^6q^2, and and conſequently its ſquare root is $= d^3q$, it follows, that when d^3q is $= 0$, that is, when the quadruple arc is equal to a ſemicircle (for in that caſe q, or the chord of the ſupplement of the quadruple arc to a ſemicircle, is $= o$, and conſequently d^3q is $= o$) one of the two quantities, $8x^4 - 8d^2x^2 + d^4$ and $8d^2x^2 - 8x^4 - d^4$, will be $= o$, and conſequently $8x^4 + d^4$ will be $= 8d^2x^2$. But when tne quadruple arc, and conſequently the ſimple arc, is infinitely ſmall, the chord x of the ſupplement of the ſimple arc to a ſemicircle will be $= d$, and conſequently $8x^4$ will be $= 8d^4$, $8d^2x^2 = 8d^4$, and $8x^4 + d^4 = 9d^4$; therefore in this caſe the quantity $8x^4 + d^4$ will be greater than $8d^2x^2$, and the exceſs will be $= d^4$. Since therefore the quantity $8x^4 + d^4$ is greater than the quantity $8d^2x^2$ when the quadruple arc is infinitely ſmall, and equal to it when the quadruple arc is equal to a ſemicircle, it follows that the quantity $8x^4 + d^4$ muſt have been greater than the quantity $8d^2x^2$ all the time the quadruple arc is increaſing from o to a ſemicircle: For as during that time x decreaſes continually (from being equal to the diameter of the circle till it becomes the chord of three half-quadrants) by infinitely ſmall degrees, 'tis evident the quantities $8x^4 + d^4$ and $8d^2x^2$ will decreaſe alſo at the ſame time by infinitely ſmall degrees, and therefore it is impoſſible the quantity $8x^4 + d^4$ ſhould from being greater, become leſs than the quantity $8d^2x^2$ before it firſt becomes equal to it, that is, before the quadruple arc becomes equal to a ſemicircle. Again, when the quadruple arc is equal to a whole circle, the chord x becomes the chord of a quadrant, and conſequently x^2 is $= \frac{d^2}{2}$, $x^4 = \frac{d^4}{4}$, $8d^2x^2 = 4d^4$, $8x^4 =$

$2d^4$, and $8x^4 + d^4 = 3d^4$; therefore at this time $8d^2x^2$ is greater than $8x^4 + d^4$, and the exceſs is $= d^4$. But it has been before ſhewn that when the quadruple arc is equal to a ſemicircle, the quantity $8d^2x^2$ is $= 8x^4 + d^4$: Therefore it follows that during all the time of the increaſe of the quadruple arc from a ſemicircle to a whole circle, or, if we ſuppoſe, (which will amount to the ſame thing,) the quadruple arc to decreaſe from a whole circle to a ſemicircle, it will follow that during the whole time of this decreaſe, the quantity $8d^2x^2$ will be greater than $8x^4 + d^4$. For as during the time of that decreaſe of the quadruple arc, the chord x increaſes continually (from being the chord of a quadrant till it becomes equal to the chord of three half-quadrants) by infinitely ſmall degrees, it is evident the quantities $8x^4 + d^4$ and $8d^2x^2$ will increaſe alſo at the ſame time by infinitely ſmall degrees, and therefore it will be impoſſible for the quantity $8d^2x^2$ from being greater to become leſs than the quantity $8x^4 + d^4$ before it firſt becomes equal to it; that is, before the quadruple arc becomes equal to a ſemicircle. Therefore as long as the quadruple arc is leſs than a ſemicircle, the equation expreſſing the relation of x and q will be $8x^4 - 8d^2x^2 + d^4 = d^3q$, and when the quadruple arc is greater than a ſemicircle, but leſs than the whole circle, that equation will be $8d^2x^2 - 8x^4 - d^4 = d^3q$.

Fourthly, Since $5d^4y - 20d^2y^3 + 16y^5$ is $= d^4p$, and conſequently $25d^8y^2 - 200d^6y^4 + 560d^4y^6 - 640d^2y^8 + 256y^{10} = d^8p^2$, it will follow by ſubſtituting $dd - xx$ for yy and $dd - qq$ for pp, that $25d^8 \times \overline{dd - x^2} - 200d^6 \times$

d^4

$\overline{d^4 - 2d^2x^2 + x^4} + 560d^4 \times$
$\overline{d^6 - 3d^2x^2 + 3d^2x^4 - x^6} - 640d^2 \times$
$\overline{d^8 - 4d^6x^2 + 6d^4x^4 - 4d^2x^6 + x^8}$
$+ 256 \times$
$\overline{d^{10} - 5d^8x^2 + 10d^6x^4 - 10d^4x^6 + 5d^2x^8 - x^{10}}$, or $25d^{10} - 25d^8x^2 - 200d^{10} + 400d^8x^2 - 200 d^6x^4 + 560d^{10} - 1680d^8x^2 + 1680d^6x^4 - 560d^4x^6 - 640d^{10} + 2560d^8x^2 - 3840d^6x^4 + 2560d^4x^6 - 640d^2x^8 + 256d^{10} - 1280d^8x^2 + 2560d^6x^4 - 2560d^4x^6 + 1280d^2x^8 - 256x^{10}$, or $d^{10} - 25d^8x^2 + 200d^6x^4 - 560d^4x^6 + 640d^8x^8 - 256 x^{10}$, is $= d^8 \times \overline{dd - qq}$, or $d^{10} - d^8q^2$, and conſequently $25d^8x^2 - 200d^6x^4 + 560d^4x^6 - 640d^2 x^8 + 256x^{10}$ is $= d^8q^2$. Therefore the ſquare root of the quantity $25d^8x^2 - 200d^6x^4 + 560 d^4x^6 - 640d^2x^8 + 256x^{10}$ is equal to d^4q. But the ſquare root of $25d^8x^2 - 200d^6x + 560 d^4x^6$ $640d^2x^8 + 256 x^{10}$ is either $16x^5 - 20d^2x^3 + 5d^4x$, or $20d^2x^3 - 16x^5 - 5d^4x$, according as $16x^5 + 5d^4x$ is greater or leſs than $20d^2x^3$. Therefore either $16x^5 - 20d^2x^3 + 5d^4x$, or $20d^2x^3 - 16x^5 - 5d^4x$, is equal to d^4q, according as $16x^5 + 5d^4x$, is greater or leſs than $20d^2x^3$. Now $16x^5 + 5d^4x$ is greater than $20d^2x^3$ as long as the quintuple arc is leſs than a ſemicircle, and leſs than $20d^2x^3$ when the quintuple arc is greater than a ſemicircle, but leſs than a whole circle; as may be thus demonſtrated. Since the ſquare root of the quantity $25d^8x^2 - 200d^6x^4 + 560d^4x^6 - 640d\, x^8 + 256x^{10}$ is either $16x^5 - 20d^2x^3 + 5d^4x$, or $20 d^2x^3 - 16x^5 - 5d^4x$, and this ſquare root is always equal to d^4q, it follows that when d^4q is $= o$, that is, when the quintuple arc is equal to a ſemicircle, (for in that caſe q, or the chord of the ſupplement of the quintuple arc to a ſemicircle,

circle, is $= o$, and conſequently d^4q is $= o$) one of the two quantities $16x^5 - 20d^2x^3 + 5d^4x$ and $20d^2x^3 - 16x^5 - 5d^4x$ will be $= o$, and conſequently $16x^5 + 5d^4x$ will be $= 20d^2x^3$. But when the quintuple arc, and conſequently the ſimple arc, is infinitely ſmall, the chord x of the ſupplement of the ſimple arc to a ſemicircle will be $= d$, and conſequently $16x^5$ will be $= 16d^5$, $20d^2x^3 = 20d^5$, $5d^4x = 5d^5$ and $16x^5 + 5d^4x = 21d^5$. Therefore in this caſe the quantity $16x^5 + 5d^4x$ will be greater than $20d^2x^3$, and the exceſs will be $= d^5$. Since therefore the quantity $16x^5 + 5d^4x$ is greater than the quantity $20d^2x^3$ when the quintuple arc is infinitely ſmall, and equal to it when the quintuple arc is equal to a ſemicircle, it follows that the quantity $16x^5 + 5d^4x$ muſt have been greater than the quantity $20d^2x^3$ all the time the quintuple arc is increaſing from o to a ſemicircle: For as x during that time decreaſes continually (from being equal to the diameter of the circle till it becomes the chord of four fifths of a ſemicircle) by infinitely ſmall degrees, it is evident the quantities $16x^5 + 5d^4x$ and $20d^2x^3$ will decreaſe alſo at the ſame time by infinitely ſmall degrees, and therefore it is impoſſible the quantity $16x^5 + 5d^4x$ ſhould from being greater become leſs than the quantity $20d^2x^3$ before it firſt becomes equal to it, that is, before the quintuple arc becomes equal to a ſemicircle. Again, when the quintuple arc is equal to a whole circle, the chord x becomes the chord of three fifths of a ſemicircle, and is therefore equal to $\frac{1+\sqrt{5}}{4} \times d$, as may be ſhewn in the following manner. When the quintuple arc is equal to a ſemicircle, we have $16x^5 - 20d^2x^3 + 5d^4x = 0$, or $16x^5 + 5d^4x = 20d^2x^3$, and conſequently $16x^4 + 5d^4 = 20d^2x^2$, and $20d^2x^2 - 16x^4 = 5d^4$, and $\frac{5d^2x^2}{4} - x^4 = \frac{5d^4}{16}$, and (ſubtracting both ſides of the equation from $\frac{25d^4}{64}$) $\frac{25d^4}{64} - \frac{5d^2x^2}{4} + x^4 = \frac{5d^4}{64}$. Therefore one of the ſquare roots of the quantity $\frac{25d^4}{64} - \frac{5d^2x^2}{4} + x^4$, that is, one of the two quantities $\frac{5d^2}{8} - x^2$ and $x^2 - \frac{5d^2}{8}$, muſt be $= \frac{\sqrt{5}}{8} \times d^2$; but the former of theſe quantities, to wit, $\frac{5d^2}{8} - x^2$, cannot be equal to $\frac{\sqrt{5}}{8} \times d^2$, becauſe that would make $x^2 = \frac{5-\sqrt{5}}{8} \times d^2$, which being leſs than $\left(\frac{5-\sqrt{4}}{8} \times d^2\text{, or } \frac{5-2}{8} \times d^2\text{, or,}\right) \frac{3d^2}{8}$, and *a fortiori* leſs than $\frac{d^2}{2}$, cannot be the true value of x^2, becauſe x^2 in the preſent caſe is the chord of four fifths of a ſemicircle or eight fifths of a quadrant, and therefore muſt be greater than the chord of a quadrant, or than $\frac{d^2}{2}$. Therefore $x^2 - \frac{5d^2}{8}$ is $= \frac{\sqrt{5}}{8} \times d^2$, and $x^2 = \frac{5+\sqrt{5}}{8} \times d^2$; that is, the chord of the ſupplement of the fifth part of a ſemicircle to a ſemicircle is $= \frac{5+\sqrt{5}}{8} \times d^2$; conſequently by the equation for the chord of the ſupplement of a double arc found in the firſt caſe of this article, to wit, $2xx - dd = dq$, or $q = \frac{2xx}{d} - d$, it follows that the chord of the ſupplement of two fifth parts of a ſemicircle, or one fifth part of a whole circle, to a ſemicircle is $= \frac{5+\sqrt{5}}{4} \times d - d = \frac{5+\sqrt{5}}{4} \times d - \frac{4d}{4} = \frac{1+\sqrt{5}}{4} \times d$. Therefore the value of x when the quintuple arc is equal to a whole circle, is $\frac{1+\sqrt{5}}{4} \times d$; therefore in this caſe x^2 is $= \frac{3+\sqrt{5}}{8} \times d^2$, $x^4 = \frac{7+3\sqrt{5}}{32} \times d^4$, $20d^2x^2 = \frac{15+5\sqrt{5}}{2} \times d^4$, $16x^4 = \frac{7+3\sqrt{5}}{2} \times d^4$, and $16x^4 + 5d^4 = \frac{7+3\sqrt{5}}{2} \times d^4 + \frac{10d^4}{2} = \frac{17+3\sqrt{5}}{2} \times d^4$, which is leſs than $\frac{15+5\sqrt{5}}{2} \times d^4$, becauſe the exceſs of 17 above 15, to wit, 2, is leſs than the exceſs of $5\sqrt{5}$ above $3\sqrt{5}$, to wit, $2\sqrt{5}$. Therefore when the quintuple arc is equal to a whole circle, $20d^2x^2$ is greater than $16x^4 + 5d^4$, and conſequently $20d^2x^3$ is greater than $16x^5 + 5d^4x$. But it has been before ſhewn that when the quintuple arc is equal to a ſemicircle, the the quantity $20d^2x^3$ is $= 16x^5 + 5d^4x$: therefore it follows that during all the time of the increaſe of the quintuple arc from a ſemicircle to a whole circle, or, if we ſuppoſe the quintuple to decreaſe from a whole circle to a ſemicircle, it will follow that during the whole time of this decreaſe the quantity $20d^2x^3$ will be greater than $16x^5 + 5d^4x$. For as, during the time of that decreaſe of the quintuple arc, the chord x increaſes continually (from being the chord of three fifths of a ſemicircle till it becomes equal to the chord of four fifths of a ſemicircle) by infinitely ſmall degrees, it is evident the quantities $16x^5 + 5d^4x$ and $20d^2x^3$ will increaſe alſo at the ſame time by infinitely ſmall degrees; and therefore it will be impoſſible for the quantity $20d^2x^3$ from being greater to become leſs than the quantity $16x^5 + 5d^4x$ before it firſt becomes equal to it, that is, before the quintuple arc becomes equal to a ſemicircle. Therefore as long as the quintuple arc is leſs than a ſemicircle, the equation expreſſing the relation of x and q will be $16x^5 - 20d^2x^3 + 5d^4x = d^4q$, and when the quintuple arc is greater than a ſemicircle, but leſs than a whole circle, the equation will be $20d^2x^3 - 16x^5 - 5d^4x = d^4q$.

134. It appears therefore, that in the equations that expreſs the relations of the chord of the ſupplement of a ſingle arc to the chord of the ſupplement of its double, triple, quadruple, and quintuple, the ſigns of the terms on the left-hand ſides of the equations, or of the terms that involve the letter x, are oppoſite, when the multiple arc is greater than a ſemicircle, but leſs than a whole circle, to what they are when the multiple arc is leſs than a ſemicircle; and the ſame may be ſhewn in all higher multiples.

135. The general reaſon of this change is that the increments or decrements of the affirmative terms, or terms marked with the ſign +, do not become equal to the contemporary incre-

ments or decrements of the negative terms, or terms marked with the ſign —, at the ſame inſtant of time that the affirmative terms themſelves become equal to the negative terms, which they muſt neceſſarily do if the affirmative terms after having become, from being greater than the negative terms, equal to them, become again immediately greater. Thus, for inſtance, in the equation $16x^5 - 20d^2x^3 + 5d^4x = d^4q$, 'tis evident that if the decrements of the quanity $16x^5 + 5d^4x$, which, during the whole time of the increaſe of the quintuple arc from o to a ſemicircle, muſt have been greater than the contemporary decrements of the quantity $20d^2x^3$ (becauſe the quantity $16x^5 + 5d^4x$ is at firſt, when the quintuple arc is infinitely ſmall, greater than the quantity $20d^2x^3$ by an exceſs equal to d^5, and, when the quintuple arc is equal to a ſemicircle, is equal to $20d^2x^3$, ſo that, during the increaſe of the quintuple arc from o to a ſemicircle, the exceſs of $16x^5 + 5d^4x$ above $20d^2x^3$ will have decreaſed continually from d^5 to o, and conſequently the decrements of the quantity $16x^5 + 5d^4x$ muſt during all that time have been greater than the contemporary decrements of the quantity $20d^2x^3$,) I ſay, if the decrements of $16x^5 + 5d^4x$ continue ſtill to be greater than the contemporary decrements of $20d^2x^3$ after the quantity $16x^5 + 5d^4x$ is become equal to $20d^2x^3$, 'tis evident the quantity $16x^5 + 5d^4x$ will become leſs than $20d^2x^3$; that is, unleſs the decrement of $16x^5 + 5d^4x$ becomes equal to the contemporary decrement of $20d^2x^3$ at the inſtant the quantities $16x^5 + 5d^4x$ and $20d^2x^3$ themſelves become equal to each other, the quantity $16x^5 + 5d^4x$ will become leſs than

$20d^2x^3$; for if, immediately after the quantity $16x^5 + 5d^4x$ becomes equal to $20d\ x^3$, the decrement of $16x^5 + 5d^4x$, inſtead of continuing greater, becomes leſs than the contemporary decrement of $20d^2x^3$, (as it muſt do to make $16x^5 + 5d^4x$, after having been equal, be again greater than $20d^2x^3$) it muſt, at the inſtant the quantity $16x^5 + 5d^4x$ is equal to $20d^2x^3$, be exactly equal to the decrement of $20d^2x^3$, becauſe till that inſtant it has been always greater than that decrement. Now that the contemporary decrements of the quantities $16x^5 + 5d^4x$ and $20d^2x^3$ cannot become equal to each other at the inſtant thoſe quantities themſelves become equal, may be ſhewn in the following manner. Put $\dot{x}$ (or the letter x with a point placed over it) for the decrement by which the chord x is diminiſhed in a very ſmall portion of time; and 'tis evident the decrement of $5d^4x$ in that time will be $5d^4\dot{x}$, that of x^3 in the ſame time will be $(= x^3 - \overline{x - \dot{x}}^3$, or) nearly $3x^2\dot{x}$, and that of x^5 will be $(= x^5 - \overline{x - \dot{x}}^5$, or) nearly $5x^4\dot{x}$; therefore the decrements of $16x^5$ and $20d^2x^3$ will be nearly equal to $5 \times 16x^4\dot{x}$ and $3 \times 20d^2x^2\dot{x}$, and conſequently the decrement of the ſum of the affirmative terms $16x^5$ and $5d^4x$ is $5 \times 16x^4\dot{x} + 5d^4\dot{x}$. Now $5 \times 16x^4\dot{x} + 5d^4\dot{x}$ cannot be equal to $3 \times 20d^2x^2\dot{x}$ at the ſame inſtant of time that $16x^5 + 5d^4x$ is equal to $20d^2x^3$, becauſe in that caſe it would follow that $5 \times 16x^4 + 5d^4$ was equal to $3 \times 20d^2x^2$ at the ſame time that $16x^4 + 5d^4$ is equal to $20d^2x^2$, which is evidently impoſſible, becauſe the former of theſe equations, to wit, $5 \times 16x^4 + 5d^4 = 3 \times 20d^2x^2$, is derived from the latter of them, to wit, $16x^4 + 5d^4 = 20d^2x^2$, by multiplying its ſeveral terms not by the ſame number (which would be neceſſary to preſerve the equality of it's oppoſite ſides) but by the ſeveral indexes of the powers of x in the original quantity $16x^5 - 20d^2x^3 + 5d^4x$. And the ſame reaſonings, 'tis eaſy to perceive, may be applied to the equations for the chords of the ſupplements of all higher multiple arcs, as well as to that of the quintuple arc.

136. *Corol.* 10. From the foregoing corollaries in general, and particularly from the laſt article of the foregoing corollary, we may conclude that whenever any quantity as z, that becomes firſt by a gradual decreaſe equal to o, and then increaſes to a certain magnitude, then decreaſes again to o, and afterwards increaſes a ſecond time, and ſo on for any number of times, is in all theſe variations of it's magnitude ſo related to two other quantities x and y that it may always be deriv'd from them by the ſame conſtructions, and conſequently is always equal to a quantity conſiſting of the ſame number of terms involving in them the ſame powers and multiples of x and y; but only differently connected together by the ſigns + and —, I ſay, when this happens, we may conclude that the ſigns of the terms of the value of the quantity z are changed into their contraries every time that quantity becomes equal to o. And by the help of this rule we may extend all the foregoing articles, concerning the methods of finding the chords, ſines, &c. of multiple arches, to the caſes of arches greater than a whole circle, and equal to any Number of whole circles, and any part of a circle, whatſoever

ſoever: Thus, for inſtance, we may conclude that the ſigns of the terms that compoſe the value of the chord of any multiple of a given arc are contrary, when that multiple arc is greater than a whole circle, but leſs than two whole circles, to what they are when the multiple arc is leſs than a whole circle; and when the multiple arc is greater than two whole circles, but leſs than three whole circles, thoſe ſigns become again the ſame as they were at firſt. But we ſhall enlarge no farther upon this ſubject.

137. *Corol.* 11. 'Tis eaſy to perceive that by this and the foregoing propoſition we are enabled to deſcribe any regular polygon in or about a circle, or to find the proportion of a ſide of any regular polygon inſcribed in, or circumſcribed about, a circle, to the diameter of the circle. For by making the multiple arc equal to a whole circle, the chord of the ſimple arc becomes a ſide of a regular polygon, inſcribed in the circle, conſiſting of the ſame number of ſides as there are parts in the multiple arc equal to the ſimple arc: And by making the multiple arc equal to a ſemicircle, the tangent of the ſimple arc becomes equal to half the ſide of a regular polygon, circumſcribed about the circle, conſiſting of the ſame number of ſides as there are parts in the multiple arc equal to the ſimple arc. Thus if I would find the length of a ſide of a regular polygon of eleven ſides inſcribed in a circle, I muſt find the equation that expreſſes the relation of the chords p and y of two arcs whereof one is to the other in the proportion of eleven to one, and making the multiple arc equal to a whole circle, and conſequently its chord $p = 0$, I muſt reſolve

the equation that reſults from that ſuppoſition, and the value of the chord y thereby obtained will, 'tis evident, be the length of a ſide of a regular polygon of eleven ſides inſcribed in the circle. In like manner, if I would find the length of a ſide of a regular polygon of eleven ſides circumſcribed about a circle, I muſt find the equation that expreſſes the relation of the tangents of two arcs whereof the one is to the other in the proportion of eleven to one, and making the multiple arc equal to a ſemicircle, and conſequently it's tangent $= o$, I muſt reſolve the equation that reſults from that ſuppoſition, and the value of the tangent of the ſimple arc thereby obtained will, it is evident, be half the length of a ſide of a regular polygon of eleven ſides circumſcribed about the circle.

138. In *Fig.* 57. the line CF exhibits all the ſecants of the arcs in the progreſſion 10°, 20°, 30°, *&c.* to ſeventy degrees, together with the ſecant of 75°; and is therefore call'd the line of ſecants. for a like reaſon, CB is call'd the line of ſines; AB the line of chords, and AE the line of tangents.

From the inſpection of this figure it appears that the difference of the ſines of two arcs decreaſes as thoſe arcs increaſe, provided their difference continues the ſame, as was demonſtrated *Prop.* 23. *Corol.* 1. but that the difference of their tangents, and likewiſe of their ſecants increaſes at the ſame time; and that at laſt when the arcs are very large, as for inſtance 75 or 80 degrees theſe differences increaſe exceeding faſt; inſomuch that an arc may be aſſigned, ſo large, or ſo near to a right angle, that if it be increaſed by any addition ever ſo ſmall, the increment of the ſecant or tangent ſhall exceed any propoſed length, ever ſo great. And thus we have a popular, but convincing, proof of ſeveral things that have been made out in a more methodical manner in the preceding propoſitions: for repreſentations of this kind that enable us to form clear conceptions of the ſeveral poſitions, magnitudes, and variations of lines in our imagination, are frequently little inferior in evidence to the more regular and preciſe reaſonings that are addreſſed to the underſtanding, at leaſt they are very fit to accompany and illuſtrate them; and they have moreover this peculiar advantage, that they greatly aſſiſt the memory in recollecting the propoſitions that have before been demonſtrated concerning them, whenever there is occaſion.

139. Hitherto we have ſuppoſed the ſines, and tangents, and other lines that relate to the meaſuring of angles, to belong to the ſame circle, we will therefore add one very eaſy propoſition concerning the relation of the ſines of different angles when taken in different circles; which with a moſt curious and uſeful theorem of *Galileo*'s concerning the locus of the vertices of any number of triangles conſtituted upon the ſame baſe, and having their ſides in the ſame ratio, and a celebrated propoſition of *Archimedes* concerning the magnitude of the area of a circle, ſhall conclude this ſecond part of theſe elements.

PROP. 27. *A Theorem.*

140. The ratio of the ſines of two different angles meaſured in different circles is compounded of

the ratio of the ſemidiameters of the circles, and the ratio of the ſines of the ſame angles taken in the ſame circle.

Let BCA, BCD (*Fig.* 58.) be the two angles; I ſay, FE the ſine of BCA in the circle FK is to BD the ſine of BCD in the circle BL in a ratio compounded of the ratios of FC to BC, and BA to BD.

For ſince FE, BA are (by the definition of a ſine) both perpendicular to CG they muſt be parallel to each other. Therefore the triangles FCE, BCA muſt be ſimilar, and FE : BA : : FC : BC. But FE is to BD in a ratio compounded of the ratios of FE to BA, and BA to BD. Therefore the ratio of FE, BD is compounded of the ratios of FC, BC, and BA, BD. QED.

141. *Corol.* 1. Hence the ratio of the ſines BA, BD, of the angles BCA, BCD, meaſured in the ſame circle, is compounded of the direct ratio of FE, BD the ſines of thoſe angles meaſured by unequal circles, and the inverſe ratio of FC, BC, the radii of thoſe circles.

Put the verſed ſine DM $= z$; EG $= x$. FC $=$ R; and BC $= r$. By *Prop.* 18, the ſine BD is a mean proportional between DM and 2 CM $-$DM; whence BDq $=$ 2 CM x DM $-$ DMq $= 2rz - zz$. For the ſame reaſon, FEq $=$ $2Rx - xx$. And by this corollary, BA : BD :: $\frac{FE}{FC} : \frac{BD}{BC}$; therefore BA is to BD as $\frac{\sqrt{2Rx - xx}}{R}$ to $\frac{\sqrt{2rz - zz}}{r}$. Let the angle BCD inſtead of being leſs than BCA, be ſo much greater than it, that it's verſed ſine DM ſhall equal EG the verſed ſine of BCA in the greater circle, or that z ſhall $= x$; and the expreſſion $\frac{\sqrt{2rz - zz}}{r}$ will be changed into $\frac{\sqrt{2rx - xx}}{r}$, and the ratio of BA to BD will be the ſame as that of $\frac{\sqrt{2Rx - xx}}{R}$ to $\frac{\sqrt{2rx - xx}}{r}$. This caſe occurs in *Helſham*'s lectures on Natural Philoſophy, lect. 9.

142. *Corol.* 2. By this figure it appears that the Arches FG, BM, ſubtending different angles BCA, BCD at the center c of the unequal circles FK, BL, are in a compound ratio of the radii FC, BC, and of the Angles they ſubtend, or the Arches BO, BM in the ſame circle. For the ratio of FG to BM is equal to the ratios of FG to BO, and BO to BM; that is, of FC to BC and BO to BM.

Hence the arches BO, BM, or the angles BCA, BCD, are to each other in the ratio of $\frac{FG}{FC}$ to $\frac{BM}{BC}$; that is, if the angles are ſmall, as $\frac{\text{chord } FG}{FC}$ to $\frac{\text{chord } BM}{BC}$ nearly. Or, if FG, BM be ſuppoſed to repreſent two Objects ſeen by an eye at C, their apparent diameters (for ſo the angles they ſubtend at the eye are call'd) will be very nearly in a ratio compounded of the direct ratio of the real diameters FG, BM, and the inverſe ratio of their diſtances FC, BC from the eye.

PROP. 28.

143. If a right line AB (*Fig.* 59 and 60) be divided into two unequal ſegments AC, CB, whereof AC is the greater, that are to each other in any propoſed ratio whatſoever, and from the extremities A, B, be drawn two lines, as AK, KB, or AF, FB, in the ſame proportion as AC, BC, meeting each other in K or F; the ſeveral interſections K, F, and C will be ſituated in the circumference of a circle, whoſe ſemidiameter is determined by the following conſtruction.

Conſtruction.

Take CP = CB; and ſay as AP : CB :: CB : BD; and D will be the center, and DC the radius of the circle in which the points K, F, *&c.* are ſituated.

Demonſtration.

Firſt of all we will ſhew that the point E, which is the laſt interſection of the lines drawn from A, B, when the angle at A is diminiſhed *ad infinitum*, lies in the circumference of this circle. Since AP : BC :: BC : BD, it follows *componendo* that, AC : CB :: CD : BD. But by the ſuppoſition, AE : BE :: AC : BC. Therefore AE : BE :: CD : BD : and *dividendo* AB : BE :: CB : BD and *permutando*, and again *dividendo*, AC : CB :: DE : BD. therefore DE : BD :: CD : BD. conſequently DE = CD, and E is in the circumference of the circle. Now take any other point as R; and producing RB till it meets the circle in S, join S, D. then we ſhall have AR : BR :: AC : CB :: CD : BD :: SD : BD. therefore (by *El.* 6, 7) ABR, SBD are ſimilar triangles; for AR is greater than BR, becauſe AC is greater than CB; therefore ang. ABR is greater than BAR, which of conſequence muſt be acute; and BSD is leſs than a right angle; becauſe it ſtands upon the line BD, which is leſs than the whole diameter CE. Therefore SB : BD :: AB : BR; whence SB x BR = AB x BD = CB x BE, becauſe BE : BD :: AB : CB by what was proved before. Therefore (by the converſe of *El.* 3. 35.) the point R is in the circumference of the circle. QED.

Note, The converſe of *El.* 3. 35 is almoſt ſelf-evident: however, to make every thing as clear as poſſible, let R be ſuppoſed not to be in the circumference of the circle; but let SB cut the circle in T or V; then by this propoſition of *Euclid*, SB x BT, or SB x BV will be = CB x BE, that is, to AB x BD; but it has been proved that SB x BR = AB x BD; therefore SB x BR = SB x BT, or SB x BV, and BR = BT, or BV, a greater to a leſs, or a leſs to a greater, which is impoſſible.

'Tis evident that what has been proved of the points E and R is equally true of all the other points K, F, I, H, G, &c.

144. *Corol.* 1. When the line AB is biſected, or when the point C coincides with the middle point O, the radius DC is infinite, and the circumference of the circle becomes a right line

perpendicular to AB in O, as in *fig.* 61. for then CB or CP being equal to AC, AP becomes = *o*, and of conſequence BD the fourth term in the proportion AP : CB :: CB : BD, muſt be infinite. Or it may be proved from this conſideration; AC : CB :: CD : BD, and AC = CB; therefore CD = BD. Now the only caſe in which the ratio of CD to BD can be a ratio of equality, is when they are both infinite, and the ratio of their difference CB to either of them is an infinitely great ratio of minority.

'Tis evident without conſidering the conſtruction of the circle, that CLKI ought to be a right line perpendicular to AB. For ſince AC = CB, it follows that AL = LB, AK = KB, AI = IB. Therefore, joining IC, the angles ACI, BCI, will be equal (by *El.* 1. 8.) and conſequently right ones. For the ſame reaſons joining KC, and LC, each of theſe lines will be perpendicular to AB in C. therefore IC, KC, LC, coincide; or the line joining the vertices L, K, I, is a right line perpendicular to AB in C.

145. *Corol.* 2. If C be taken nearer and nearer to B, the circle will continually grow leſs and leſs, and at laſt when C and B coincide will vaniſh entirely: for in this caſe CB decreaſes and AP increaſes; therefore the ratio of AP to CB, and conſequently of CB to BD, increaſes; therefore BD decreaſes as well as CB. conſequently the radius CD, or CB + BD, decreaſes. When CB = *o*, BD will = *o*, and CD, and conſequently the circle of which it is the radius, will vaniſh.

Pla. VI.

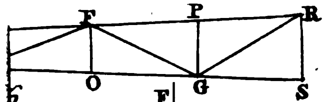
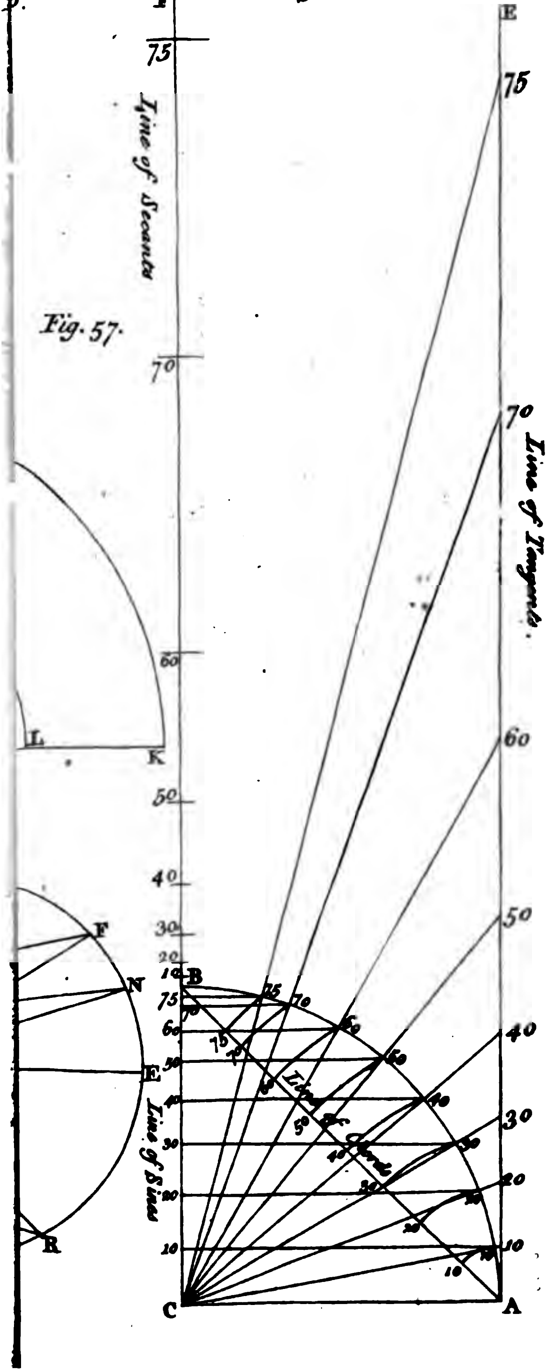

F
P
R
O
G
S
6.

Fig. 57.


F
E
75
70
60
50
40
30
20
10
Line of Secants
Line of Tangents
L
K
F
N
E
R
B
Line of Sines
Line of Chords
C
A

146. *Corol.* 3. If a right line AB is divided into two ſegments AC, CB, that are to each other in any propoſed ratio whatſoever, and (taking CP = CB, and making AP : CB :: CB : BD) a circle be deſcribed with the center D, and radius DC; the ratio of any two lines AK, KB, or AF, FB, or AI, IB, drawn from any point K, F, or I, in the circumference of the circle to A, B, the extremities of the given line, will be the ſame with that of the ſegments of AC, CB.

Previouſly to the following propoſition, it will be convenient to make the following obſervations.

Obſ. 1. The area of a regular polygon, as BDFHKMOQB, (*fig.* 62.) circumſcribed about a circle, is equal to a right-angled triangle, whoſe baſe is equal to the perimeter of the polygon, and altitude to the radius of the circle. For by drawing the lines HR, KR, MR, *&c.* from the ſeveral angles of the polygon to the center of the circle, the polygon may be divided into the ſeveral triangles HRK, KRM, *&c.* which have the ſides of the polygon for their baſes, and the radius of the circle for their common perpendicular altitude, and therefore (by *El.* 6. 1.) it is equal to a triangle whoſe baſe is equal to the ſum of all theſe baſes, or of all the ſides of the polygon, and whoſe perpendicular altitude is equal to the radius of the circle, or to a right-angled triangle whoſe baſe is the perimeter of the polygon, and height the radius of the circle.

Obſ. 2. The area of a regular polygon, as *bdfhkmoqb*, (*fig.* 62.) inſcribed in a circle, is equal to a right-angled triangle whoſe baſe is equal to the perimeter of the polygon, and altitude to R*i*, or the coſine of half the arch *kb* ſubtended by one of the ſides of the polygon. For the inſcribed polygon *bdfhkmoqb* may be divided into the ſeveral triangles *b*R*k*, *k*R*m*, *&c.* whoſe baſes are the ſides of the polygon, and their common altitude equal to R*i*.

Obſ. 3. If a regular polygon be circumſcribed about a circle, the number of its ſides may be ſo far increaſed that the area of the polygon ſhall exceed the area of the circle by an exceſs that is leſs than any aſſign'd area whatſoever. For if BDFHKMOQB (*fig.* 62.) is a regular polygon circumſcribed about the circle RAEIN, and *bdfhkmoqb* a polygon ſimilar to the former inſcribed in that circle, the proportion of the former polygon to the latter will (by *El.* 6. 20.) be equal to that of (KH*q* to *kbq*, or KI*q* to *kiq*, or) RI*q* to R*iq*, or of the ſquare of the radius to the ſquare of the coſine of half the arch ſubtended by a ſide of the inſcribed polygon, which proportion, 'tis evident, may, by increaſing the number of ſides in the polygons, be made to approach as near as we pleaſe to a ratio of equality: therefore by increaſing the number of ſides in the polygons the circumſcribed polygon may be made to exceed the inſcribed polygon by as ſmall a quantity as we pleaſe, and conſequently, *a fortiori*, it may be made to exceed the circle, which is always greater than the inſcribed polygon, by an exceſs that is leſs than any aſſigned area whatſoever.

PROP. 29.

148. The area of every circle is equal to a right-lined triangle whoſe baſe is the circumference of the circle, and altitude the radius.

Let AEIN (*fig.* 62.) be a circle whoſe center is R, and radius AR; and STV (*fig.* 63.) a right-angled triangle whoſe baſe TV is equal to the circumference of the circle AEIN, and altitude TS to the radius AR. I ſay, the area of the circle AEIN is equal to the triangle STV. For if they are not equal, let us firſt ſuppoſe the triangle to be the greater, and the difference to be equal to the ſpace X. About the circle AEIN circumſcribe the regular polygon BDFHKMOQB, the number of whoſe ſides is ſo great that the exceſs of it's area above the area of the circle AEIN ſhall be leſs than the ſpace X; which, by *Obſ.* 3. it is poſſible to do; and, it is evident, the circumſcribed polygon BDFHKMOQB will be leſs than the triangle STV. But, by *Obſ.* 1. the circumſcribed polygon is equal to a right-angled triangle, whoſe baſe is equal to the perimeter of the polygon, and altitude to the radius of the circle: therefore this triangle is leſs than the triangle STV. But, becauſe by the lemma in *Art.* 31, the perimeter of the circumſcribed polygon is greater than the circumference of the circle, and conſequently the baſe of the former triangle is greater than the baſe of the latter triangle, and their altitudes are equal, it follows that the former triangle muſt be greater than the latter: but it has been been before ſhewn to be leſs than t latter triangle: therefore it is both greater a leſs than the latter triangle at the ſame tim which is impoſſible; therefore the triangle ST is not greater than the circle AEIN. QED.

Secondly, Let the triangle STV be ſuppoſed to leſs than the circle AEIN, and let the difference equal to X. Inſcribe in the circle AEIN the regu polygon *bdfhkmoqb*, the number of whoſe ſides is great that the exceſs by which the circle AEI exceeds its area ſhall be leſs than the ſpace X which, it appears from *El.* 12. 2. is poſſible to b done: and it is evident the inſcribed polygo *bdfhkmoqb* will be greater than the triangle STV. But, by *Obſ.* 2. the polygon *bdfhkmoqb* is equal to a right-angled triangle whoſe baſe is equal to the perimeter of the polygon, and its altitude to the coſine R*i*; therefore this triangle is greater than the triangle STV. But it is alſo evidently leſs than the triangle STV, becauſe its baſe, being equal to the perimeter of the inſcribed polygon, is leſs than the baſe of the triangle STV, which is equal to the circumference of the circle, and its altitude R*i* is alſo leſs than the altitude of the triangle STM, or than the radius of the circle: therefore it is both leſs and greater than the triangle STV, at the ſame time; which is impoſſible. Therefore the triangle STV is not leſs than the circle AEIN. But it was before ſhewn that it was not greater than that circle; therefore it is equal to it. QED.

149. *Note*, If we make uſe of the conciſer, though leſs accurate, language of infinitely ſmall quantities, the demonſtration of the foregoing

propoſition may be reduced to this; that if the number of the ſides of a regular polygon circumſcribed about a circle be increas'd, and conſequently, their magnitude diminiſh'd, *ad infinitum*, the perimeter and area of the polygon will coincide with the circumference and area of the circle reſpectively; and conſequently by *Obſ.* 1. the area of the circle will be equal to a right-angled triangle, whoſe baſe is equal to the circumference, and its altitude to the radius, of the circle. A like demonſtration of this propoſition might be deduced from the conſideration of the inſcribed polygons, by the help of *Obſ.* 2.

150. *Corol.* 1. Hence 'tis evident that the area of a circle is equal to a rectangle contained under its ſemicircumference and, radius or, ſemidiameter.

151. *Corol.* 2. Hence it appears likewiſe that to ſquare the circle, or to find the proportion of its area to the ſquare of the diameter in numbers, it is neceſſary firſt to know the proportion of the diameter to the circumference; that is the quadrature of the circle, and the finding the proportion of the diameter to the circumference in numbers, are one and the ſame problem, or are mutually dependant upon each other. 'Tis commonly ſaid that the diameter and circumference of a circle are incommenſurable to each other, and conſequently that an accurate quadrature of the circle is impoſſible: but this I have never ſeen demonſtrated, though it ſeems probable from this conſideration, ſuggeſted by Dr. *Barrow*, to wit, " That the ſides of regular polygons in-" ſcribed in, and circumſcribed about, a circle

" grow

" grow more and more involved with ſurd quan-" tities, or quantities incommenſurable to the, " diameter, as their number is increaſed;' whence it ſeems reaſonable to conjecture that the circumference of the circle, which is the laſt value of the perimeters, or ſums of the ſides, of the circumſcrib'd and inſcrib'd polygons when the number of their ſides is infinite, ſhould be infinitely involved with ſurd quantities, and therefore be incommenſurable to the diameter. However the proportion of the diameter to the circumference of a circle has been expreſſed in numbers to a prodigious degree of exactneſs by ſeveral authors, and particularly by Mr. *Euler*, who in his treatiſe of infinite ſeriefes, as has carried his numbers to upwards of an hundred and twenty places of decimal figures. Some of the moſt convenient methods of computing theſe numbers will be ſhewn in *Part* 3. of theſe *Elments*.

152. *Corol.* 3. From this propoſition *Galileo* has deduced the following curious property of the circle, to wit, that its area is greater than the area of any regular polygon of equal perimeter with itſelf. This he demonſtrates by ſhewing that the area of a circle is a mean proportional between the area of any regular polygon whoſe perimeter is equal to its circumference, and the area of a polygon ſimilar to the former, circumſcribed about the circle, in the following manner. Let P (*fig.* 64.) be any regular polygon whoſe perimeter is equal to the circumference of the circle AEIN (*fig.* 62.) and BDFHKMOQB be a polygon ſimilar to the polygon P circumſcribed about that circle. Then, ſince the area of the circum-

ſcribed polygon is equal to a right-angled triangle, whoſe baſe is equal to it's perimeter and altitude to the radius of the circle, and the area of the circle is equal to a right-angled triangle, whoſe baſe is equal to its circumference and altitude to its radius, it follows (by *El.* 6. 1.) that the area of the circumſcribed polygon is to the area of the circle as the baſe of the former triangle is to the baſe of the latter triangle, that is, as the perimeter of the circumſcribed polygon is to the circumference of the circle, and therefore as the perimeter of the circumſcribed polygon to the perimeter of the polygon P. But becauſe the polygon P is ſimilar to the circumſcribed polygon, their proportion to each other will be duplicate of the proportion of their perimeters. Therefore the proportion of the circumſcribed polygon to the polygon P is duplicate of its proportion to the circle AEIN; and conſequently that circle will be a mean proportional between the polygon circumſcribed about it and the polygon P. And therefore, as the circumſcribed polygon is always greater than the circle, whatever be the number of the ſides, it follows that the circle will always be greater than the polygon P, which is iſoperimetral to it, of what number of ſides ſoever that polygon conſiſts. QED.

153. *Galileo* demonſtrates further that of two regular polygons circumſcribed about a circle, that which has the greater number of ſides is the leſſer; and from thence he deduces a proof, that of two regular polygons that are iſoperimetral to each other, that which has the greater number of ſides is the greater.

154. The former of theſe particulars he demonſtrates in the following manner. Let AD be half the ſide of any regular polygon circumſcribed about the circle AOF (*fig.* 65.) whoſe center is O, and AC half the ſide of any other regular polygon conſiſting of a greater number of ſides than the former, circumſcribed about the ſame circle, and let *n* be the number of ſides in the former polygon, and *m* the number of ſides in the latter polygon. Draw the ſecants OC, OD, cutting the circle AOF in G and F; and with O as a center and OC as a radius deſcribe the circular arch ICE cutting OA (produced) in I and OD in E. Then 'tis evident that AC will be leſs than AD, and conſequently OC, or OE, than OD, and the triangle DOC will be = the ſector EOC + the ſpace CED. therefore the proportion of the triangle DOC to the triangle COA is the ſame with that of the ſum of the ſector EOC and ſpace CED to the triangle COA; and therefore, as the ſector COI is greater than the triangle COA, the proportion of the triangle DOC to the triangle COA will be the ſame with the proportion of the ſum of the ſector EOC, the ſpace CED, and ſome third quantity as y, to the ſector COI; that is, DOC is to COA as the ſector EOC increas'd by the quantity DCE + y to the ſector COI. But the ſector EOC is to the ſector COI in the ſame proportion as the ſector FOG is to the ſector GOA; therefore the quantity EOC + DCE + y is to the ſector COI as the ſector FOG increas'd by ſome quantity which call z, to the ſector GOA. therefore the triangle DOC is to the triangle COA as FOG + z is to GOA. Therefore, *componendo*, we have DOA : COA ::

:: FOA $+ z$: GOA; and, *permutando*, DOA : FOA $+ z$ :: COA : GOA. Therefore DOA is to FOA alone as COA increas'd by ſome quantity, which call v, to GOA. Therefore $2n$ times the triangle DOA, or the area of the whole polygon of which AD is half a ſide, is to $2n$ times the ſector FOA, or the area of the whole circle AOF, as $2m$ times the quantity COA $+ v$, that is, as $2m$ times the triangle COA increas'd by $2m$ times v, is to $2m$ times the ſector GOA, or the area of the whole circle AOF. therefore the area of the polygon of which AD is half a ſide is equal to the area of the polygon of which AC is half a ſide increaſed by the quantity $2m \times v$, or is greater than that polygon. QED.

Note, This property of the circumſcribed polygons might alſo have been deduced from *Art.* 147. *Obſ.* 3. as the reader will eaſily perceive.

155. The ſecond of the foregoing particulars, to wit, " that, if there be two regular polygons " whoſe perimeters are equal to each other, that " which has the greater number of ſides will be " greater than the other," may be eaſily deduced from the foregoing articles in the following manner.

Let there be any two iſoperimetral polygons which call A and B; and of theſe polygons let A be that which has the greater number of ſides, we are to ſhew that the polygon A is greater than the polygon B. Let AOF (*fig.* 65.) be a circle whoſe circumference is equal to the perimeter of either of theſe polygons, and AOC and AOD two polygons, reſpectively ſimilar to the polygons A and B, circumſcribed about the circle AOF. Then, ſince by *Art.* 152, the circle AOF is a mean proportional between the circumſcribed polygon AOC and the polygon A, and alſo between the circumſcribed polygon AOD and the polygon B; and, by *Art.* 154, the circumſcribed polygon AOC is leſs than the circumſcribed polygon AOD; it follows the polygon A will be greater than the polygon B. QED.

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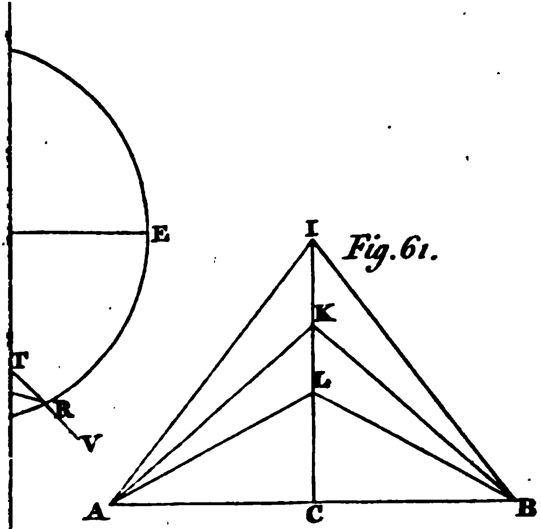
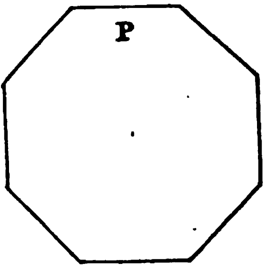
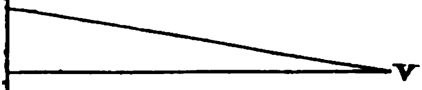
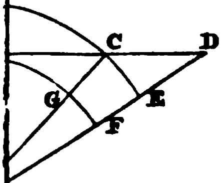


Fig. 61.

Fig. 64.







J. Mynde fc.

PART the THIRD.

Concerning the fluxions of ſines, tangents, ſecants, verſed ſines and chords.

156. LET the arc RL of the ſemicircle RLC (*fig.* 66.) be conceived to be generated by the motion of a point moving from R to L round the center C with an uniform velocity, ſo as to deſcribe equal arches in equal times. Draw the radius C*r* at right angles to CR; the tangent RG, and the line KN, each of an indefinite length; and the line MF equal to RL the diameter of the circle. And let the five lines C*r*, RC, MF, RG, and KN, be all ſuppoſed to have begun to be generated at the ſame inſtant of time as the arch RL, by the motion of as many points moving from C towards *r*, from R towards C, from M towards F, from R towards G, and from K towards N, with ſuch velocities that at every following inſtant of time the generated portion of C*r* ſhall be equal to the ſine of the arc generated at that inſtant, the generated portion of RC to its verſed ſine, that of MF to its chord, that of RG to its tangent, and that of

KN to its ſecant. Thus when the point R has deſcribed the arch RA, the firſt point ſhall have deſcribed the line Cϲ equal to DA its ſine; the ſecond point ſhall have deſcribed RD its verſed ſine; the third point ſhall have deſcribed the line MH equal to RA its chord; the fourth point ſhall have deſcribed RP its tangent; and the fifth point ſhall have deſcribed the line Ky equal to its ſecant CP: when the point R has deſcribed the arch Ra, the other points ſhall have deſcribed the lines Cα, RB, Mb, RT, and KΔ, which are reſpectively equal to its ſine, verſed ſine, chord, tangent, and ſecant. Theſe lines, it muſt be obſerved, do not all continue to increaſe till the arch becomes a ſemicircle; but ſome of them, to wit, the ſine, tangent, and ſecant, arrive at their limit, or at their greateſt magnitude, when the arch RA becomes equal to a quadrant.

157. Now it is evident that, the arch being ſuppoſed to increaſe with an uniform velocity, the velocities with which all the other lines are generated will not be uniform, but continually varying in every inſtant of time. Thus the tangent and ſecant flow with velocities that are continually increaſing, and the ſine with a velocity that is continually decreaſing. Conſequently the velocities of theſe ſeveral lines at any propoſed inſtant of time are not proportional to the increments they receive in any finite time whatſoever beginning at that inſtant.

158. Nevertheleſs by continually diminiſhing theſe increments we may make their proportions approach continually nearer and nearer to thoſe of their velocities in the firſt inſtant of their generation: for theſe velocities are accurately pro-

portional to the lines that would be generated by them in any given time, if the velocities had continued uniform during that time; and the leſs theſe increments are taken, the leſs will be the difference of the velocities in each of them at the beginning and end of the time in which they are deſcribed, or the nearer will they approach to being deſcribed with uniform velocities. And as this diminution of the increments may be continued *ad infinitum*, and the difference of the velocities in each increment at the beginning and end of the time of its deſcription be thereby diminiſhed *ad infinitum* likewiſe, it follows that the proportion of theſe ſeveral increments to each other may be made to approach infinitely near to the ratio of the velocities at the beginning of the time of their deſcription, that is, ſo near as to differ from it by leſs than any aſſigned proportion whatſoever.

159. Therefore the ratio of the velocities of theſe ſeveral lines at any aſſigned inſtant of time is equal to the *limit* of the ratio of the contemporary increments that begin to be generated at that inſtant, or to that ratio to which their proportion may by diminiſhing them continually be made to approach ſo near that its difference therefrom ſhall be leſs than any aſſigned proportion whatſoever. This limit is frequently call'd the *ultimate ratio* of the increments, becauſe it is the proportion they bear to each othea in the *laſt* inſtant of their exiſtence, juſt as, by the diminution aforeſaid, they are reduced to a ſtate of evaneſcence, or are paſſing into nothing. But as this conception is ſomewhat ſubtle, the reader will do well, as often as the phraſe, *ultimate ratio*

occurs, to carry in his mind the idea of the limit above defined.

160. To illuſtrate what has here been ſaid concerning theſe increments in general, we will conſider particularly the increments PT and ϵα of the tangent and ſine. Now 'tis evident that as PT and ϵα may be diminiſh'd *ad infinitum*, the difference of the velocities in P and T, and alſo that of the velocities in ϵ and α, may thereby be diminiſhed *ad infinitum* likewiſe, and be made to bear to the velocities in P and ϵ reſpectively a leſs than any aſſigned proportion whatſoever; ſo that theſe increments may be made to approach as near as we pleaſe to being deſcribed with uniform velocities. Conſequently their ratio to each other may be made to approach as near as we pleaſe to the *ratio* of the velocities in P and ϵ. Whence 'tis evident that the *ratio* of the velocities in P and ϵ is equal to the limit of the *ratio* of the increments PT and ϵα. For the ſame reaſon the *ratio* of the velocities in P and A, and that of the velocities in ϵ and A, are reſpectively equal to the limits of the *ratios* of PT to A*a*, and ϵα to A*a*.

161. The ſame is evidently true of any other lines whatſoever, belonging to any other figure beſides a circle, that are generated by the motion of points in the manner above-deſcribed; to wit, that the proportion which the velocities of the points that generate any two of them bear to each other at any aſſigned inſtant of time is equal to the limit of the *ratio* of the contemporary increments that begin to be generated at that inſtant.

Quantities generated in this manner are commonly call'd *fluents*, that is, flowing or variable quantities; and the velocities wherewith they flow are call'd their *fluxions*.

To determine the proportions of theſe fluxions, or velocities, in all the lines deſcribed in the preceding articles, and to expreſs them in known finite terms is the deſign of the following propoſition; in the *Corol.* 11 to which it muſt be obſerved that by the velocities of the ſine, chord, and ſecant, are meant the velocities of the points that generate the lines C*r*, MF, KN, reſpectively equal to them.

PROP. 30. *A Theorem.*

162. The fluxion of the tangent of any arc is to the fluxion of the arc itſelf, as the ſquare of the ſecant of that arc to the ſquare of the radius.

The *ratio* of the velocities in the points P and A (*fig.* 66.) of any tangent and arc RP and RA is, as has been before obſerv'd, equal to the limit to which the *ratio* of the contemporary increments PT and A*a* approaches as thoſe increments are continually diminiſh'd. With the center C and radius CP deſcribe the circular arc PV cutting CT in V. Then it muſt be obſerv'd that ſince the *ratio* of PT to A*a* is always equal to the ſum of the *ratios* of PT to PV and PV to A*a* during the whole time of the diminution of thoſe increments, the limit of the *ratio* of PT to A*a* muſt equal to the ſum of the limits of the other two *ratios*. Now by diminiſhing PT 'tis evident that the mixtilinear triangle TVP approaches to a ſtate of ſimilitude and equality with a rectilinear triangle formed by drawing a right line from P perpendicular to CT; and as this diminution may be continued *ad infinitum*, this approach may be continued *ad infinitum* likewiſe. Conſequently the limits of the *ratios* of the ſides of the mixtilinear triangle are equal to the limits of the *ratios* of the correſponding ſides of the rectilinear triangle, or of the triangle CRT, which is ſimilar to it; therefore the limit of the *ratio* of PT to PV is equal to the limit of the *ratio* of CT to CR; that is, to the ratio of CP to CR. But the *ratio* of PV to A*a* is always equal to the *ratio* of CP to CR, they being ſimilar arches of the circles deſcribed with thoſe radii; therefore its limit is equal to the ſame. Conſequently, the ſum of the limits of the ratios of PT to PV and PV to A*a*, or the limit of the ratio of PT to A*a*, is equal to twice the ratio of CP to CR; that is, the ratio of the velocities in P and A, or of the fluxions of the tangent and arc, is equal to that of the ſquare of the ſecant CP to the ſquare of the radius CR. Q.E.D.

This demonſtration may be expreſs'd more briefly, though with leſs preciſeneſs, in the following manner.

The ratio of the fluxions of the tangent and arc is, by what has been before obſerved, equal to the ultimate ratio of the contemporary increments PT, A*a*. With the center C, and radius CP deſcribe the circular arc PV cutting CT in V. Now 'tis evident that, when PT is infinitely diminiſhed, CT coincides with CP, the arc PV coincides with its ſine, or with a line drawn from P perpendicular to CT, and conſequently the triangle TPV becomes a rectilinear and rectangular triangle ſimilar to CTR; therefore PT is to PV as CT, or CP, to CR. But PV is to A*a* as CP to CR, they being ſimilar arches of the circles deſcribed with thoſe radii. Therefore *ex æquo* PT : Aa :: CPq : CRq. QED.

163. By comparing theſe two demonſtrations, or rather theſe two ways of expreſſing the ſame demonſtration, together, and by attentively conſidering what has been ſaid in *Art.* 158, 159, and 160, 'tis preſumed that the ſenſe in which the phraſes, "*ultimate ratio, ultimate coincidence* "*of a curve with a right line, ultimate ſimilitude of a mixtilinear to a rectilinear area, and* "*the like,*" are to be underſtood, will be ſufficiently evident: for which reaſon I thought it unneceſſary to obſerve the ſame caution in expreſſing the demonſtrations of the following *Corollaries*, as in the firſt demonſtration of the preceeding theorem, and accordingly have expreſſed them in the ſame looſe manner as the latter of the foregoing demonſtrations; apprehending that the long circumlocutions neceſſary to expreſs them accurately would only tend to perplex and embarraſs the ſentences, and thereby render the connexion between the ſeveral ſteps of each demonſtration leſs eaſily perceptible.

Corol. 1. Hence the tangent of any arc is greater than the arc itſelf, as was before ſhewn in *Art.* 32.

For 'tis evident that the velocity of the point that generates the tangent equals at firſt, and ever after exceeds, the velocity of the point that generates the arc.

164. *Corol.* 2. The velocity of the arc is to the velocity of the ſine, as the ſecant is to the radius, or the radius to the coſine.

For the proportion of theſe velocities is equal to the ultimate *ratio* of A*a* to C*a* or *ba*. Now the triangle A*ba* is ultimately ſimilar to CPR, becauſe CRP, A*ba*, are right angles, and PCR (= CA*b* = CA*a* − *a*A*b* = (*El.* 3. 18.) a right angle — *a*A*b*) = (*El.* 1. 32.) *ba*A. therefore A*a* : *ba* :: CP : CR :: CA : CD.

Corol. 3. The fluxion of the tangent is to the fluxion of the ſine as the cube of the ſecant to the cube of the radius. For, by the propoſition, PT : A*a* :: CP*q* : CR*q*; and by *Corol.* 2. A*a* : *ba* :: CP : CR. therefore *ex æquo*, PT : *ba* :: CP^3 : CR^3.

For inſtance, the ſecant of 60° is double of the radius: therefore the velocity of the point that generates the tangent is to the velocity of that which generates the arc, at the inſtant that the arc of 60° is compleatly deſcrib'd, as 4 to 1; and to the velocity of the ſine at that inſtant as 8 to 1: ſo that, if theſe velocities were all to continue uniform, the point that deſcribes the tangent would run over a ſpace of 8 inches, and the point that deſcribes the ſine a ſpace of 1 inch, while the point that deſcribes the arch generated a line 2 inches long.

165. *Corol.* 4: The fluxion of the ſecant is to the fluxion of the arc as the rectangle under the ſecant and tangent to the ſquare of the radius. For theſe fluxions are in the ultimate proportion of $y\triangle$ to A*a*, or of VT to A*a*. But VT is to A*a* in a compound *ratio* of VT to PT, and PT to A*a*; that is, of RP to CP, and CP*q* to CR*q*; that is, in the *ratio* of CP*q* × RP to CR*q* × CP, or of CP × RP to CR*q*.

166. *Corol.*

Corol. 5. The fluxion of the tangent is to the fluxion of the ſecant as the ſecant is to the tangent. For PT : VT :: CP : RP.

166. *Corol.* 6. The fluxion of the arc is to the fluxion of the verſed ſine as the ſecant to the tangent, or as the radius to the ſine. For when the arc is of any magnitude as RA leſs than a quadrant, the triangle A*ba* is ultimately ſimilar to CRP, or CDA; and in like manner when the arc is of any magnitude as R*r*E (*fig* 66.) greater than a quadrant, the triangle E*ze* form'd by the differences of the arch, ſine, and verſed ſine, is ultimately ſimilar to the triangle CL*p* form'd by the radius, tangent, and ſecant of R*re*, or to the triangle CES form'd by the radius, ſine, and coſine of the ſame arch. Therefore A*a* : A*b* = DB :: CP : RP :: CA : AD; and E*e* : *ze* = SO :: C*p* : L*p* :: CE : ES. that is, (by *Art.* 159.) the *ratio* of the velocities of the arch and verſed ſine is during the deſcription of the whole ſemicircle equal to the *ratio* of the ſecant and tangent, or radius and ſine.

167. *Corol.* 7. The fluxion of the arc is to the fluxion of the chord as the chord to the ſine, or as the diameter to the chord of the ſupplement to a ſemicircle. In *fig.* 67. let the arches RA, RE, the former of which is leſs, and the latter greater than a quadrant, be increaſed by the differences A*a*, E*e*. Draw the ſines AD, ES; the chords RA, R*a*, RE, R*e*; and the chords AL, EL, of the ſupplements of RA, RE, to a ſemicircle; and ſuppoſe the increments A*a*, E*e*, to be diminiſhed *ad infinitum*. 'Tis evident that each of the triangles

AR*f*, ER*g* becomes ultimately iſoſceles; becauſe, when the angle AR*f* vaniſhes, the angle A*f*R becomes the complement of *f*AR to two right angles, and therefore equal to *f*AR which is itſelf a right angle by *El.* 3. 31. And for the ſame reaſon, when the angle ER*g* vaniſhes, the angle E*g*R becomes equal to *g*ER. therefore RA is ultimately equal to R*f*, and RE to R*g*; and *fa*, *ge*, are ultimately equal to the differences of the chords A, R*a*, and RE, R*e*, or to H*b*, P*p*, the differences of the lines MH, M*b*, and MP, M*p*, reſpectively equal to thoſe chords. Now becauſe the arches A*a*, E*e* coincide ultimately with their chords, the mixtilinear triangles A*fa*, and E*ge*, will coincide ultimately with the rectilinear triangles form'd by the lines A*f*, *fa*, and the chord of *a*A, and E*g*, *ge*, and the chord of E*e*; and therefore become ultimately ſimilar to the triangles L*f*R, and L*g*R, in which the angles L*f*R, and L*g*R, are equal to A*fa*, E*ge*, becauſe vertical to them, and the angles RL*f*, RL*g* are equal to *fa*A, *ge*E, becauſe they ſtand upon the ſame arches RA, RE. Therefore ultimately, $Aa : fa = Hb :: RL : Lf$, and $Ee : ge = Pp :: RL : Lg$, or (becauſe L*f* is ultimately equal to LA, and $Lg = LE$) $Aa : Hb :: RL : LA$, and $Ee : Pp :: RL : LE$; that is, (by *Art.* 159.) the *ratio* of the velocities of the arch and chord is during the deſcription of the whole ſemicircle equal to the *ratio* of the diameter to the chord of the ſupplement, or (becauſe the triangles RLA, RDA, and likewiſe the triangles RLE, RSE, are ſimilar, whence $RL : LA :: RA : AD$, and $RL : LE :: RE : ES$) to the *ratio* of the chord to the ſine.

168. *Corol.* 8. Hence the fluxion of the verſed ſine is to the fluxion of the chord as twice the chord is to the diameter. For the fluxion of the verſed ſine being to that of the arc as the ſine to the radius, and the fluxion of the arc to that of the chord as the chord to the ſine, it follows that the fluxion of the verſed ſine is to the fluxion of the chord as the rectangle under the ſine and chord to the rectangle under the radius and ſine, or as the chord to the radius, or twice the chord to the diameter.

169. As the verſed ſine is always a third proportional to the diameter and chord, this corollary is a demonſtration of the following theorem.

If two variable lines begin to be generated from nothing at the ſame inſtant of time, and continue to increaſe together till the firſt of them becomes equal to a certain given line, and their relation during the whole time of their generation be ſuch that the ſecond variable line is always a third proportional to the foreſaid given line, and the firſt variable line; the velocity of the point that generates the ſecond variable line will always be to the velocity of the point that generates the firſt variable line as twice the firſt variable line to the given line.

170. This theorem is true as well when the variable lines increaſe *ad infinitum*, as when they are ſuppoſed to increaſe only till they become equal to a given line, as may be ſhewn by the contemplation of a few very ſimple properties of the parabola.

Let DC*c* (*fig.* 68.) be a parabola, whoſe focus is B, vertex D, and principal parameter AB. From any two points C, *c*, that are ſituated near to each other in the perimeter of the parabola, draw the ordinates CP, *cp* to the axis DP; and from C (the point neareſt to the vertex D) draw C*e* parallel to the axis cutting *cp* in *e*, and the tangent CG cutting the axis in G. Conceive this parabola to have been generated by the motion of the ordinate CP moving from the vertex D parallel to itſelf upon the axis DP, and at the ſame time increaſing in length in ſuch a manner that its extremity C ſhall generate the parabolic curve DC*c*; and let CS be a line generated by the motion of the point S, and always equal to CP. Then 'tis evident the ordinate CP, and abſciſs DP, or the line CS, and the abſciſs DP, will be two ſuch variable lines as were deſcribed in the theorem; for they both begin to be generated from nothing at the ſame inſtant of time, and the latter is (by *Simſon*'s Conic Sections, B. 1. Prop. 12.) always a third proportional to the parameter AB and the former of them. Now the velocity with which CP moves forwards parallel to itſelf, or the velocity of the point P that generates the abſciſs DP, is to the velocity with which the line CP increaſes in length, or to the velocity of the point S that generates the line CS, which is always equal to CP, in the ultimate *ratio* of the contemporary increments P*p*, and S*s*, or C*e* and *ce*; that is, (becauſe the triangle C*ce* becomes ultimately ſimilar to GCP) as GP to CP, or (by *Simſon*, B. Prop. 10.) as 2 DP to CP, or 2 CP to AB. And this is equally true whether CP and DP be leſs, or greater than AB.

171. From this theorem it follows *e converſo*, that, whenever, by conſidering the velocities of two variable lines that both begin to be gene-

rated from nothing at the ſame inſtant of time, and afterwards increaſe together, we find that the velocity of the ſecond is to the velocity of the firſt during the whole time of their generation as twice the firſt is to a certain given line, we may conclude that the ſecond variable line is a third proportional to the given line and to the firſt variable one.

172. This converſe and the preceding theorem are ſo cloſely and ſo evidently connected, that when either of them has been demonſtrated it ſeems needleſs to demonſtrate the other. But ſome are of opinion that the more elegant method would be to demonſtrate the converſe, and deduce the theorem from it; becauſe this is the order in which they naturally occur to us in the ſolution of mathematical problems, it being uſually neceſſary for that purpoſe, not from the knowledge of the relations of the quantities concerned in them to determine the relations of their fluxions, but from the relations of the fluxions to determine thoſe of the quantities generated by them. Which of theſe methods of demonſtration is the eaſier I will not pretend to determine; although the former, to wit, that by which the relations of the fluxions are determined from the relations of the quantities, ſeems to be ſo, and, perhaps for that reaſon, has generally been made uſe of by moſt writers on fluxions: but it is certain, that both methods are equally accurate and juſt.

173. *N. B.* Hitherto, as often as we have had occaſion to conſider the velocity with which a line increaſes when the line itſelf is continually varying its poſition, as the chord, ſine, and ſecant

in the circle, and the ordinate to the axis in the parabola, in which caſes the points that generate theſe lines have a complicated motion; we have drawn lines given in poſition which we have ſuppoſed to be generated by the motion of points moving with ſuch velocities that the portions generated by them in any aſſigned times whatſoever ſhall always be reſpectively equal to ſuch portions of the lines they repreſent as are generated in the ſame times; and we defined the velocities with which the lines that vary their poſition increaſe to be the velocities of the points that generate the lines reſpectively equal to them. In the following propoſitions the ſame things are to be underſtood as before concerning the velocities of any lines that vary their poſition while they are increaſing in length, and the ſame lines are ſuppoſed to be drawn reſpectively equal to them; but it is thought unneceſſary actually to deſcribe them.

Scholium.

174. From the preceding corallaries it may not be amiſs to deduce a few obſervations concerning the variations of the velocities of the points that generate the lines belonging to a circle.

And *firſt*, 'tis evident that the *ratio* of the velocities of the points that generate the tangent and arc is at the firſt inſtant of their generation a *ratio* of equality, and afterwards increaſes *ad infinitum*, as the arc increaſes or approaches to a quadrant, ſo that if any *ratio*, how great ſoever be aſſigned, the arc may be taken ſo near to a quadrant that this *ratio* ſhall exceed it.

Secondly, The *ratio* of the velocities of the points that generate the arc and ſine, is at firſt a *ratio* of equality, and afterwards increaſes *ad infinitum*, as the arc approaches nearer and nearer to a quadrant, ſo that if any *ratio*, how great ſoever be aſſigned, the arc may be taken ſo near to a quadrant, that this *ratio* ſhall exceed it.

Thirdly, The *ratio* of the velocities of the points that generate the ſecant and arc is in the firſt inſtant of their generation an infinitely great *ratio* of minority, and afterwards decreaſes *ad infinitum*. as the arc increaſes, ſo as to become a *ratio* of equality when the arc has attained a certain magnitude; to wit, ſuch a magnitude that the rectangle under the ſecant and tangent ſhall be equal to the ſquare of the radius, which will be found to be 38° 11′. After this it becomes a *ratio* of the contrary kind, or a *ratio* of majority, and increaſes *ad infinitum* as the arc approaches nearer and nearer to a quadrant; ſo that, if any *ratio*, how great ſoever be aſſigned, the arc may be taken ſo near to a quadrant that this *ratio* ſhall exceed it.

Fourthly, The *ratio* of the velocities of the points that generate the arc and verſed-ſine is in the firſt inſtant of their generation an infinitely great *ratio* of majority, and afterwards decreaſes *ad infinitum* as the arc approaches nearer and nearer to a quadrant, ſo as to become a *ratio* of equality when the arc becomes equal to a quadrant; then it increaſes again as the arc from being a quadrant approaches nearer and nearer to a ſemicircle, and becomes an infinitely great *ra-*

tio of majority when the arc is equal to a ſemicircle.

Fifthly, The *ratio* of the velocities of the points that generate the arc and chord is in the firſt inſtant of their generation a *ratio* of equality, and afterwards increaſes *ad infinitum* as the arc approaches nearer and nearer to a ſemicircle, ſo that, if any *ratio*, how great ſoever be aſſigned, the arc may be taken ſo near to a ſemicircle, that this *ratio* ſhall exceed it.

The truth of theſe obſervations will be evident by conſidering that the *ratios* of the velocities mentioned in them are reſpectively equal to the *ratios* of, the ſquare of the ſecant to the ſquare of the radius, the ſecant to the radius, the rectangle under the tangent and ſecant to the ſquare of the radius, the radius to the ſine, and the diameter to the chord of the ſupplement of the arc to a ſemicircle, by *Art.* 162, 164, 165, 166, 167; which *ratios* vary in the manner deſcribed in theſe obſervations.

175. From the preceding obſervations it is evident, that if the arc be ſuppoſed, as above, to be deſcribed with an uniform velocity, the velocities wherewith the other lines are deſcribed will vary in the following manner.

Firſt, The velocity of the tangent will be at firſt equal to that of the arc, and afterwards will increaſe *ad infinitum* as the arc approaches to a quadrant, ſo that if any velocity, how great ſoever be aſſigned, the arc may be taken ſo near to a quadrant that the velocity of the tangent ſhall exceed it.

Secondly, The velocity of the ſine will be at firſt equal to that of the arc, and afterwards will decreaſe.

decreaſe *ad infinitum* as the arc approaches to a quadrant, ſo that if any velocity, how ſmall ſoever be aſſign'd, the arc may be taken ſo near to a quadrant, that the velocity of the ſine ſhall be exceeded by it.

Thirdly, The velocity of the ſecant will be at firſt infinitely ſmall, or (to expreſs this more cautiouſly) if any velocity, how ſmall ſoever, be aſſign'd, the arc may be taken ſo ſmall that the velocity of the ſecant ſhall be exceeded by it; afterwards this velocity will increaſe *ad infinitum* as the arc approaches to a quadrant, ſo that if any velocity, how great ſoever, be aſſign'd, the arc may be taken ſo near to a quadrant that the velocity of the ſecant ſhall exceed it.

Fourthly, The velocity of the verſed ſine will be at firſt infinitely ſmall, or, if any velocity, how ſmall ſoever, be aſſign'd, the arc may be taken ſo ſmall, that the velocity of the verſed ſine ſhall be exceeded by it; afterwards it will increaſe as the arc approaches to a quadrant, and be exactly equal to the velocity of the arc when the arc is become a quadrant; then it decreaſes again *ad infinitum* as the arc increaſes from nothing to a ſemicircle, ſo that if any velocity, how ſmall ſoever be aſſigned, the arc may be taken ſo near to a ſemicircle that the velocity of the verſed ſine ſhall be exceeded by it.

Fifthly, The velocity of the chord will be at firſt equal to that of the arc, ſine, or tangent, and afterwards will decreaſe *ad infinitum* as the arc increaſes from nothing to a ſemicircle, ſo that if any velocity, how ſmall ſoever be aſſigned, the arc may be taken ſo near to a ſemicircle,

circle, that the velocity of the chord ſhall be exceeded by it.

176. *Note*, Although in the preceding propoſition and corollaries the arc has been ſuppoſed to have been generated by a point moving with an uniform velocity, yet this ſuppoſition was by no means neceſſary. For the proportions of the fluxions of the arcs, ſines, verſed ſines, chords, tangents, and ſecants, will be juſt the ſame whether we ſuppoſe the arch to have been generated with an uniform velocity or not; becauſe they are the ſame with the ultimate *ratios* of the contemporary increments of thoſe lines, which ultimate *ratios* cannot be at all affected by ſuch a ſuppotion. But, as that ſuppoſition ſeem'd to be the moſt natural and convenient, and the fitteſt to convey clear notions of the relative magnitudes of thoſe ſeveral lines, it was therefore thought proper to make uſe of it.

177. The preceding articles afford a good oppotunity of preventing a miſtake which Beginners ſometimes fall into concerning the limits of the *ratios* of variable quantities. They are apt to imagine that when two quantities either increaſe or decreaſe together *ad infinitum*, the limit to which their proportion approaches muſt be a *ratio* of equality. Now, though this be ſometimes the caſe, yet it is far from being generally true: For it often happens that the *ratio* of two quantities that increaſe or decreaſe together *ad infinitum*, either increaſes *ad infinitum*; or approaches to ſome fix'd *ratio*, different from a *ratio* of equality, as its limit; or

elſe continues always the ſame during all the variations of the quantities themſelves. Of ſome of theſe caſes the laſt propoſition and its corollaries afford us proper inſtances.

The ratio of the abſciſs of the axis of a parabola to the ordinate to the axis (being equal to that of the ordinate to the parameter) increaſes *ad infinitum*, as the abſciſs and ordinate either increaſe or decreaſe *ad infinitum*, as the abſciſs and ordinate either increaſe or decreaſe *ad infinitum*.

The *ratio* of the tangents of ſimilar arcs of two concentric circles continues always equal to the ſame invariable *ratio* of the radii of thoſe circles; as thoſe tangents either increaſe, or decreaſe, *ad infinitum*, by increaſing the arc to a quadrant, or diminiſhing it to nothing.

As to the *ratio* of the ſecant and tangent of a circular arc, it does indeed approach as the arc increaſes to a *ratio* of equality as its limit, as is evident by conſidering that it is always equal to the *ratio* of the radius and ſine. But the reaſon of this is not that they both increaſe indefinitely as the arc approaches to a quadrant, but becauſe the tangent, which is the leſſer of the two, always receives greater increments than the ſecant, by *Prop.* 30. *Corol.* 5. In like manner the *ratio* of the tangent and ſine of a circular arc approaches, as the arc decreaſes, to a *ratio* of equality as it's limit, not ſimply, becauſe they both decreaſe, but becauſe during their decreaſe, the tangent, which is the greater, loſes much greater decrements than the ſine: I ſay, much greater; becauſe if the decrements of the tangent exceeded thoſe of the ſine only in the proportion of the tangent to the ſine, the *ratio* of theſe lines would continue the ſame during the whole time of their decreaſe. But this is not the caſe; for the ſmall decrements of the tangent and ſine are to each other in the triplicate *ratio* of the tangent and ſine themſelves, the latter *ratio* being equal to that of the ſecant to the radius, and the former (by *Prop.* 30. *Corol.* 3.) to that of the cube of the ſecant to the cube of the radius. And in general, the cauſe upon which the magnitude of the *ratio* of two quantities that vary *ad infinitum*, and of the limit of this *ratio*, depends, is not ſimply the indefinite increaſe or decreaſe of the quantities themſelves, but the proportion of their contemporary increments or decrements.

The uſual conciſe manner of expreſſing the foregoing inſtances is this: The abſciſs and ordinate to the axis of a parabola become infinite in an infinite *ratio* of majority, and vaniſh in an infinite *ratio* of minority. The tangents of ſimilar arcs of two concentric circles become infinite in the *ratio* of the radii of thoſe circles, and likewiſe vaniſh in the ſame *ratio*. The ſecant and tangent of a circular arc become infinite in a *ratio* of equality. Other inſtances may be taken, from the tangent and arc, which vaniſh in a *ratio* of equality; the arc and chord, which vaniſh likewiſe in a *ratio* of equality; the arc and ſine, which vaniſh in a *ratio* of equality; the arc and verſed ſine, which vaniſh in an infinite *ratio* of majority; and the arc and variable part of the ſecant, or exceſs of the ſecant above the radius, which vaniſh likewiſe in an infinite *ratio* of majority.

178. *Corol.* 9. If there be two arcs, AO, AP, (*fig.* 69.) whoſe difference OP is equal to RS the difference of two greater arcs AR, AS; the *ratio* of the tangents of the two greater arcs will not always be greater than the *ratio* of the tangents of the two leſſer, but leſs in ſome caſes and greater in others. Theſe caſes are thus determined.

Firſt, If the greater of the two leſſer arcs as much exceeds 45°, as the leſſer falls ſhort of it, or, in other words, if one of the leſſer arcs is the complement of the other to a quadrant; the *ratio* of the greater tangents will be greater than the *ratio* of the leſſer tangents.

And if the greater of the two leſſer arcs does more exceed 45° than the leſſer falls ſhort of it; likewiſe, if both the leſſer arcs exceed 45°; or, (to comprehend both theſe caſes in one expreſſion) if the ſum of the two leſſer arcs be greater than a quadrant; the *ratio* of the greater tangents will be greater than the *ratio* of the leſſer tangents.

Secondly, If the greater of the two greater arcs as much exceeds 45°, as the leſſer falls ſhort of it, or if one of the greater arcs is the complement of the other to a quadrant; the *ratio* of the greater tangents will be leſs than the *ratio* of the leſſer tangents.

And if the greater of the two greater arcs does leſs exceed 45° than the leſſer falls ſhort of it; likewiſe, if both the greater arcs are leſs than 45°; or (to comprehend both theſe caſes in one expreſſion) if the ſum of the two greater arcs is leſs than a quadrant; the *ratio* of the greater tangents will be leſs than the *ratio* of the leſſer tangents.

Thirdly, If the two leſſer arcs are as much leſs than 45° as the two greater arcs are greater than 45°; that is, if the firſt or leaſt arc is the complement of the fourth or greateſt, and (conſequently) the ſecond arc of the third, to a quadrant; the *ratio* of the two greater tangents will be equal to the *ratio* of the two leſſer.

If the two leſſer arcs be greater than the complements of the two greater; the *ratio* of the greater tangents will be greater than the *ratio* of the leſſer tangents.

And if the two leſſer arcs be leſs than the complements of the two greater; the *ratio* of the greater tangents will be leſs than the *ratio* of the leſſer.

179. Theſe determinations may be proved diſtinctly as follows:

Firſt, Let A*b*, A*d*, A*f*, A*a*, (*fig.* 69.) be four arcs, of which the two latter are the greater; let the difference *bd* be equal to the difference *fa*; and let A*b* be as much leſs as A*d* is greater than the arc A*t* of 45°. I ſay, AL is to AK in a greater proportion than AH to AG.

This will be evident, if we can ſhew that HL is to GK in a greater proportion than AH to AG. Now let the whole quadrantal arc be ſuppoſed to have been generated by an uniform motion, as in *Art.* 156. Then ſince the velocity with which HL begins to be deſcribed, or which it has in the point H, is to the velocity of the arc as CH*q* to CA*q*, by *Art.* 162; and the velocity of the arc is to the velocity with which GK begins to be deſcribed, or which it has in the point G, as CA*q* to CG*q*; it follows that the velocity of HL in H is to the velocity of GK in G as CH*q* to CG*q*. And the ſucceeding velocities of HL are greater than the correſponding velocities of GK in the proportion of the ſquares of the intermediate ſecants between CH and CL to the ſquares of the intermediate ſecants between CG and CK. Now, if the velocity of HL was everywhere throughout its generation in the ſame *ratio* to the correſponding velocity of GK as it is at firſt; 'tis evident, the ſpace HL would be to the ſpace GK in the *ratio* of theſe firſt velocities, or of CH*q* to CG*q*. But becauſe (by *Prop.* 24. *Corol.* 3.) the *ratio* of the velocities, or ſquares of the ſecants, is continually increaſing, the ſpaces deſcribed by them, to wit, HL and GK, muſt be to each other in a greater *ratio* than that of the firſt velocities, or of CH*q* to CG*q*. But (by *Prop.* 16.) CH*q* : CG*q* :: Cg^q : Cb^q, that is, (becauſe the arches A*b*, T*d*, and A*d*, T*b*, and conſequently their ſines *gb*, C*b*, and *bd*, C*g*, are equal) :: C*g* × *bd* : C*b* × *gb* :: (by *Prop.* 17.) AH : AG. Therefore HL is to GK in a greater proportion than AH to AG. QED.

If A*b* and A*d* be increaſed ſo, that the point *b* ſhall either be nearer to *t* than it was before, or than *d* is to *t*; or, that it ſhall paſs through *t* and lie on the ſame ſide of it as *d*; the ſines *bd*, *gb* will be increaſed, and at the ſame time C*g*, C*b* will be diminiſhed. Therefore (*Prop.* 24. *Coroll.* 2.) the *ratio* of C*g*, C*b* will be greater than the *ratio* of *bd*, *gb*. Therefore the *ratio* of Cg^q to Cb^q, or of CH*q* to CG*q*, will be greater than that of C*g* × *bd* to C*b* × *gb*, or of AH to AG. therefore *à fortiori* the *ratio* of HL to GK (which now as before muſt be greater than that of CHq to CGq) will be greater than the *ratio* of AH to AG. conſequently AL will be to AK in a greater proportion than AH to AG. QED.

Secondly, Let the two leſſer arcs be AR, AS, and the two greater Ab, Ad, of which Ab is as much leſs as Ad is greater than 45°. I ſay, AH is to AG in a leſs proportion than AF to AE. This will be evident if we can ſhew that FH is to EG in a leſs proportion than AH to AG. For if this be true, there is a certain quantity which being added to FH will make it bear the ſame proportion to EG as AH does to AG. Let that quantity be x. Then ſince AH : AG :: FH $+ x$: EG, it follows (*El.* 5. 19.) that AF $- x$: AE :: AH : AG. Therefore AH is to AG in a leſs *ratio* than AF to AE.

Now becauſe the *ratio* of the velocities with which FH, EG are deſcribed (being that of the ſquares of the ſecants) is continually increaſing, the lines themſelves muſt be in a leſs *ratio* than their laſt or greateſt velocities: that is, FH is to EG in a leſs *ratio* than CHq to CGq, or Cg^q to Cb^q, or (becauſe Cg = bd, and Cb = gb) than Cg × bd to Cb × gb, or (by *Prop.* 17.) than AH to AG. QED.

If the arcs Ab, Ad be diminiſhed, ſo that the point d ſhall either be nearer to t than it was before, or than b is to t; or that it ſhall paſs through t and lie on the ſame ſide of it as b; the ſines bd, gb will be diminiſhed, and Cg, Cb increaſed. Therefore (by *Prop.* 24. *Corol.* 2.) the *ratio* of bd to gb will be greater than that of Cg to to Cb. Therefore the *ratio* CHq to CGq,, or of Cg^q to Cb^q, will be leſs than the *ratio* of Cg ×

bd to $Cb \times gb$, or of AH to AG. Therefore *a fortiori* the *ratio* of FH to EG (which now as before muſt be leſs than that of CHq to CGq) will be leſs than that of AH to AG. conſequently AH will be to AG in a leſs proportion than AF to AE. QED.

Thirdly, Let the two leſſer arcs be AO, AP, and the two greater Av, Ay; and let AO = Ty, and AP = Tv. I ſay, the *ratio* of the tangents of Ay, Av, will be equal to the *ratio* of the tangents of AP, AO.

For $Cr = OV$; $Cp = PM$; $ry = CV$; and $pv = CM$. Therefore $ry \times Cp : pv \times Cr :: CV \times PM : CM \times OV$; and (by *Prop.* 17.) Tang; Ay : Tang; Av :: Tang; AP : Tang; AO. QED.

Or it may be deduced from *Art.* 64. for Tang; Ay : Tang; Av :: *co*T; Av : *co*T; Ay, by that article. But *co*T; Av = Tang; Tv = Tang; AP; and *co*T; Ay = Tang; Ty = Tang; AO. Therefore Tang; Ay : Tang; Av :: Tang; AP : Tang; AO. QED.

If the two leſſer arcs be AR, AS, greater than AO, AP, the complements of the two greater arcs Ay, Av; the *ratio* of the tangents of Ay, Av, will be greater than the *ratio* of the tangents of AS, AR. For the *ratio* of the tangents of AP, AO is greater than that of the tangents of AS, AR, ſo long as AS, AR, are either both leſs than 45°, or if one of them exceeds and the other falls ſhort of 45°, the exceſs of the greater above 45° is equal to, or leſs than, the defect of the leſſer below 45°, by what has been ſhewn in the ſecond part of this *Corollary*. But the *ratio* of the tangents of Ay, Av, is equal to that of the tangents of AP, AO; conſequently it is

greater

greater than that of the tangents of AS, AR in the caſes juſt now mentioned.

If the arcs AS, AR are either both greater than 45°, or one greater and the other leſs, and the exceſs of the greater above 45° be greater than the defect of the leſſer below it, then by the firſt part of this *Corollary*, the *ratio* of the tangents of *Ay*, *Av*, will be greater than the *ratio* of the tangents of AS, AR. QED.

In like manner it may be ſhewn, that if the two leſſer arcs be leſs than AO, AP, the complements of the two greater arcs *Ay*, *Av*; the *ratio* of the tangents of *Ay*, *Av* will be leſs than the *ratio* of the tangents of the leſſer arcs. QED.

180. The demonſtration here given of this *Corollary* is, I am ſenſible, extremely tedious, and ſomewhat indirect; and accordingly I ſhould not have made uſe of it had I not been unable, after the moſt diligent endeavours, to find a ſimpler and more elegant method of proving it, that was equally regular and accurate; for it would hardly have been ſufficiently ſo to make uſe of the following conſiderations, which are the only avenues I know of to the diſcovery of the truth of it.

The *ratio* of the tangents of two arcs whoſe difference is given, is equal to that of the rectangle under the ſine of the greater and coſine of the leſſer to the rectangle under the ſine of the leſſer, and coſine of the greater arc. This *ratio* is infinite at both extremities of the quadrant, to wit, when the leſſer arc is infinitely ſmall, and the greater equal only to their difference, and when the greater is equal to a quadrant: at all other times it is a finite *ratio*; and at equal diſtances from the middle of the quadrant it is of the ſame magnitude, the ſines of the arcs when greater than 45° being equal to their coſines when as much leſs than 45°, and their coſines in the former caſe to their ſines in the latter: therefore its leaſt magnitude muſt be when the arcs lie on different ſides of 45°, and one of them as much exceeds it as the other falls ſhort of it; conſequently when the arcs depart from 45°, and become either leſs or greater, this *ratio* muſt increaſe. I ſay, theſe conſiderations though true, have hardly, in the looſe form they here appear in, a ſufficient degree of evidence to paſs for a geometrical demonſtration: and to demonſtrate them in a regular manner would be very troubleſome.

181. The determinations might indeed be expreſſed in a conciſer manner than above, and would have been ſo, had I not apprehended that it was proper to enunciate them in ſuch terms as were adapted to the demonſtrations that were to follow. However, it may not be amiſs in this place to expreſs them in another manner more eaſy to be remembered, after having firſt ſhewn that they take in all the caſes that can happen. The caſes are theſe:

Firſt, Three of the arcs may be greater, and the fourth leſs than 45°.

2dly, Three of them may be leſs, and the fourth greater than 45°.

3dly, All the four may be greater than 45°.

4thly, All the four may be leſs than 45°.

5thly, Two of them may be greater, and two leſs than 45°.

6thly, Three of them may be greater than 45°, and the fourth exactly equal to it.

7thly, Three of them may be leſs than 45°, and the fourth exactly equal to it.

8thly, Two of them may be greater than 45°, the next equal to it, and the leaſt leſs.

9thly, Two of them may be leſs than 45°, the next equal to it, and the greateſt greater.

The firſt of theſe caſes may be divided into three parts, to wit; the exceſs of the leaſt arc but one, or third arc, above 45° may be either equal to, greater than, or leſs than, the defect of the fourth below it. If it be equal, or greater, this caſe is determined by the firſt part of this *Corollary*; if leſs, by the third part.

The ſecond caſe may alſo be divided into three parts, to wit; the exceſs of the fourth, or greateſt arc above 45° may be either equal to, greater than, or leſs than, the defect of the third arc below 45°. If it be equal, or leſs, this caſe is determined by the ſecond part of this *Corollary*; if greater, by the third part.

The third caſe is determined by the firſt part, the fourth caſe by the ſecond part; the fifth caſe by the third part; the ſixth caſe by the firſt part; the ſeventh caſe by the ſecond part; the eighth and ninth caſes by the third part.

Now the determinations may be expreſſed as follows.

Firſt, If the ſum of the two leſſer arcs be either equal to, or greater than a quadrant; the *ratio* of the greater tangents will be greater than the *ratio* of the leſſer tangents.

2dly, If the ſum of the two greater arcs is either equal to, or leſs than, a quadrant; the *ratio* of the greater tangents will be leſs than the *ratio* of the leſſer tangents.

3*dly*, If the two leſſer arcs be greater than, equal to or leſs than; the complements of the two greater arcs; that is, if the leaſt arc be greater than, equal to or leſs than, the complement of the greateſt to a quadrant, and (conſequently) the ſecond of the third; the *ratio* of the greater tangents will be accordingly greater than, equal to, or leſs than, the *ratio* of the leſſer tangents.

182. If ſtill greater brevity is required, the reader may eaſily obſerve that the firſt and ſecond of theſe determinations are in fact contained in the laſt; which is therefore the only one that need be remembered.

183. *Corol.* 10. In ſome of the foregoing articles, the infinitely ſmall line PV (*fig.* 66.) has been of great uſe to us in determining the relations of TP, A*a*, and TV, the fluxions of the tangent, arc, and ſecant, to each other. 'Tis natural therefore on account of its connection with the circle, and will be of conſiderable uſe in ſome propoſitions hereafter to be delivered, to inquire into the nature and properties of the variable line of which PV is the fluxion, or which is generated by the motion of a point that begins to move at the ſame inſtant of time that the arc RA and tangent RP begin to be generated, and continues to move during the whole time of their generation with a velocity that is to the velocity of the point that generates the tangent RP as PV to PT, or as the radius CA to the ſecant CP. Now this variable line is the logarithm of the

ratio of the ſum of the ſecant CP and tangent RP to the radius CA in a logarithmic curve whoſe ſubtangent is the radius CA; or it is equal to the portion of the axis of ſuch a logarithmic curve intercepted between two ordinates, whereof the leſſer is equal to the radius CA, and the greater to the ſum of the ſecant CP and tangent RP. In order to demonſtrate this aſſertion, it will be neceſſary to give ſome account of logarithms and the logarithmic curve, and explain a few of their principal properties.

A
DISSERTATION
ON THE
Nature and Uſe of LOGARITHMS.

Definition of logarithms.

184. IF there be two ſets of quantities ſo related to each other, that the quantities in the ſecond ſet are proportional to, or are meaſures of, the *ratios* of the quantities in the firſt ſet to each other, the quantities in the ſecond ſet are called the logarithms of the *ratios* of the quantities in the firſt ſet: and this, whether the quantities of the ſecond ſet be quantities of the ſame kind with thoſe of the firſt ſet, or of any other kind whatſoever.

185. That the foregoing definition of logarithms may be the better underſtood, it will not be amiſs to give ſome explanation of what is meant

Fig. 67.

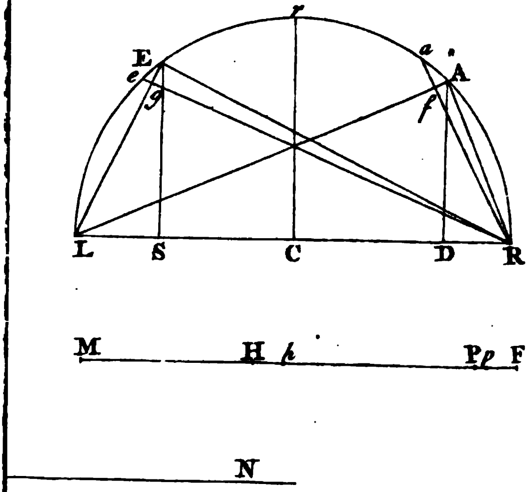
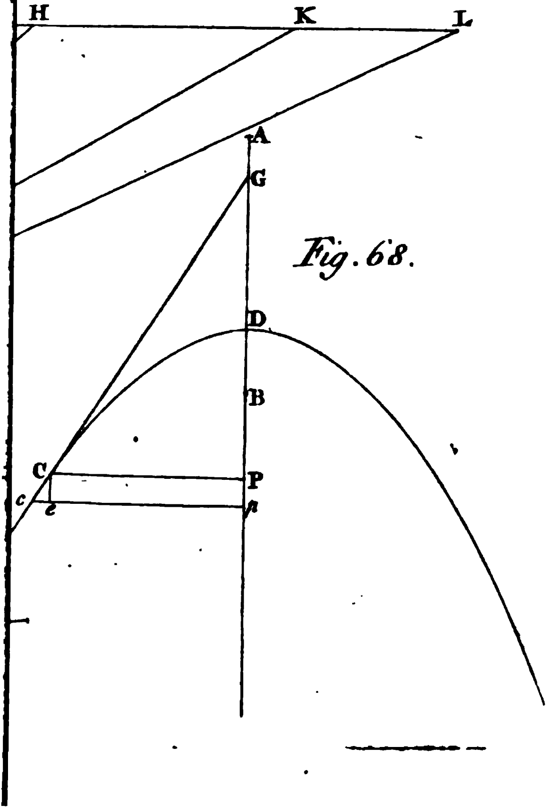


Fig. 68.



meant by the proportions of *ratios* one to another, and to ſhew how they are derived from the idea of proportion in other quantities. Now this will appear from the following conſiderations.

Of the nature of proportion in general, and particularly of the proportion of ratios *one to another.*

IN the firſt place we get the idea of proportion in general, by comparing together two quantities of the ſame kind, as ſuppoſe lines, (for as they are ſimpler and more eaſily conceived than any other quantities, they are the fitteſt to inſtance in) and conſidering how many equal parts of the one being added together would make a line equal to the other: if the lines are of ſuch lengths that either of them being laid over the other, would exactly cover it without either falling ſhort of or going beyond it, the lines are ſaid to be equal, or their proportion to be a *ratio* or proportion of equality. If the lines are of ſuch lengths that if one of them be laid upon the other, it will either go beyond it or fall ſhort of it, they are ſaid to be unequal, or their proportion to be a *ratio*, or proportion, of inequality; and the particular quantity of their inequality is found by inquiring how many times the greater line, or that which goes beyond the other, contains either the whole leſſer line, or a certain aliquot part of it, or part which being repeated a certain number of times would be equal to the whole leſſer line. This is all that we can poſſibly inquire concerning their proportion; and thereerefore when this is known, their proportion is ſaid to be accurately determined.

But 2*dly*, It has been diſcovered by mathematicians that there may exiſt ſome lines, or other quantities, whoſe proportion to each other is ſo ſubtle, that it cannot be expreſſed in numbers, or which are ſo related to each other, that into whatever number of equal parts one of them be divided, it will be impoſſible for any other number of thoſe parts to be exactly equal to the other line. Theſe lines, as they admit of no common meaſure, or line that will meaſure them both, or whereof both the lines are multiples, are ſaid to be incommenſurable to each other. Of this nature are the ſide and diagonal of a ſquare, and a variety of other lines that occur in geometrical inquiries. Nevertheleſs, the proportions of theſe lines may always be expreſſed in numbers to a greater degree of exactneſs than any that can be aſſigned, though never with perfect exactneſs; theſe proportions therefore are known, as far as their nature will permit them to be known, when we are able to expreſs them in numbers to as great a degree of exactneſs as we pleaſe.

3*dly*, Having thus got the idea of proportion in lines and other very ſimple quantities which we are daily obliged to contemplate, we may without much difficulty apply it to quantities that are leſs familiar to us, and amongſt others to proportions themſelves: for wherever we can frame to ourſelves any notion of equality between two things of the ſame kind, and of adding ſeveral of theſe equal quantities together, 'tis evident we ſhall always be able to apply the ideas of proportion mentioned in the preceding paragraphs to theſe quantities. Thus, for inſtance, if we define two angles to be equal when, if the one be laid over the other, it will exactly cover it, or will neither fall within it nor without it; and further call one angle double, or triple, or any other multiple, of another, when, by drawing one, two, or any other number of lines from the angular point between the two lines that contain it, we may conſtitute two, three, or any other number of angles equal to each other and to the other angle; I ſay, if we lay down theſe two definitions (for they are but definitions), we ſhall have as clear an idea of the proportions of two angles one to another as we can have of the proportion of two lines, and ſhall underſtand any aſſertion made concerning the proportion of the angles, as for inſtance, an aſſertion that the one angle is to the other as 5 to 4, as well and as fully as we do the like aſſertion concerning the proportion of the lines. In like manner, if the proportion of any two quantities A and B be defined to be equal to the proportion of two other quantities C and D, when, if B be divided into any number whatſoever of equal parts, and D be divided likewiſe into the ſame number of equal parts, the third quantity C conſiſts of the ſame number of the equal parts of the fourth quantity D as the firſt quantity A does of the like equal parts of the ſecond quantity B, or C is the ſame multiple of one of the equal parts of D as A is of one of the like equal parts of B; or, if A is incommenſurable to B, and C to D, when, if B and D be divided into any number whatſoever of equal parts, the third quantity C is of an intermediate magnitude between the ſame multiples of one of the equal parts of the fourth quantity D as the firſt quantity A is of one of the like equal parts of the ſecond quantity B; or (according to *Euclid's* definition, *El.* 5. *def.* 5, which Dr. *Saunderſon* has demonſtrated in his *Algebra*, *Art.* 262. to be equivalent to the foregoing definition) when the quantities A, B, C and D are ſo related to each other that, if any equimultiples whatſoever be taken of A and C the firſt and third quantities, and any other equimultiples whatſoever be taken of B and D the ſecond and fourth quantities, the equimultiple of A can never exceed, equal, or fall ſhort of, the equimultiple of B, but at the ſame time the equimultiple of C ſhall reſpectively exceed, equal, or fall ſhort of, the equimultiple of D; I ſay, if we define the four quantities A, B, C, D to be proportional, or the proportion of A to B to be equal to the proportion of C to D, when theſe quantities are related to each other in the manner here deſcribed, and if we further call one *ratio*, or proportion, double, or triple, or any other multiple, of another proportion, when by inſerting between the terms of the former proportion one, two, or any other number of quantities, of an intermediate magnitude between thoſe terms, we can conſtitute, or make to ariſe, two, three, or any other number of proportions that are all equal to each other and to the latter proportion; we ſhall have as clear an idea of the proportion of two *ratios*, or proportions, as we can have of the proportion of two lines, or any other quantities, one to another, and ſhall as fully underſtand any aſſertions that can be made concerning them. Thus, to make uſe of the ſame inſtance we before mentioned for angles, if it be aſſerted that the proportion of two given *ratios* is that of 5 to 4, the meaning of this aſſertion will be this; to wit, that if the greater of the two *ratios* be divided into five equal parts, or there are four intermediate proportionals inſerted between it's terms, ſo that there are thereby made to ariſe five *ratios* all equal to each other, and the leſſer of the two *ratios* is divided into four equal parts, or there are three intermediate proportionals inſerted between it's terms, ſo that there are thereby made to ariſe four *ratios* all equal to each other, each of theſe four equal *ratios* that ariſe out of the leſſer of the two given *ratios* is equal to each of the five equal *ratios* that ariſe out of the greater of the two given *ratios*.

Thus we ſee what is meant by the proportions of *ratios*, and therefore cannot be at a loſs to underſtand the foregoing definition of logarithms, or to conceive a ſet of quantities proportional to the *ratios* of the terms of another ſet of quantities, or whoſe proportions to each other are reſpectively the ſame with the proportions of the *ratios* of the correſpondent quantities of the other ſet one to another.

Of the aſymptotic areas of hyperbolas.

186. From the foregoing definition of logarithms it follows, that the mixtilinear areas intercepted between the aſymptote, curve, and ordinate of an hyperbola, are logarithms of the *ratios* of the ordinates that bound them: For it

has been demonſtrated by the writers on the conic ſections, and particularly by the Marquis de l'Hoſpital, *Book* 5, *Art.* 224, 225, that, when the *ratio* of any two ordinates is equal to the *ratio* of two other ordinates, the hyperbolic area, or trapezium, intercepted between the two former ordinates is equal to the hyperbolic area, or trapezium, intercepted between the two latter ordinates, from whence 'tis an eaſy conſequence that where the *ratios* of the ordinates are unequal, they are proportional to the areas contained between them. But as this is a very curious and important property of the hyperbola, and intimately connected with the ſubject of the preſent corollary, it may not be diſagreeable to the reader to ſee it demonſtrated in more than one manner; I ſhall therefore give the following proof of it.

187. The propoſition to be demonſtrated is this: If VA*d* (*fig.* 70.) be an hyperbola whoſe vertex is A, center C, and aſymptotes CT, CD, and on the aſymptote CD we take CD to CE as CF to CG, and draw the ordinates D*d*, E*e*, F*f*, G*g*, parallel to the other aſymptote CT, and meeting the hyperbola in *d*, *e*, *f*, *g*; the hyperbolic area, or trapezium, DE*ed* will be equal to the hyperbolic area, or trapezium, FG*gf*.

For if they are not equal, let DE*ed* be ſuppoſed to be greater than FG*gf*, and let their difference be equal to the rectangle Z. Between CD and CE inſert any number of intermediate proportionals CH, CK, and between CF and CG inſert the ſame number of intermediate proportionals CL, CM; ſo that CD ſhall be to CH as CH to CK, and CH to CK as CK to CE, and CF ſhall be to CL as CL to CM, and CL to CM as CM to CG; and conſequently (becauſe the *ratio* of CD to CE is equal to that of CF to CG) that each of the three former *ratios* ſhall be equal to each of the three latter *ratios*. Through the points D, H, K, E, F, L, M, G, draw the ordinates D*d*, H*b*, K*k*, E*e*, F*f*, L*l*, M*m*, G*g*, parallel to the aſymptote CT, and meeting the hyperbola in *b*, *d*, *k*, *e*, *f*, *l*, *m*, *g*; from *a* draw *en* parallel to CD cutting K*k* in *n*; from *k* draw *kp* parallel to CD cutting H*b* and E*e* in *p* and P; from *b* draw *b*N parallel to CD cutting D*d*, K*k*, E*e*, in N, *v*, R; and from *d* draw *d*S parallel to CD cutting H*b*, K*k*, E*e*, in *t*, *r*, S; in like manner from *g* draw $g\alpha$ parallel to CD cutting M*m* in α; from *m* draw $m\delta$ parallel to CD cutting L*l* and G*g* in δ and μ; from *l* draw $l\theta$ parallel to CD cutting F*f*, M*m*, G*g*, in θ, *o*, *q*; and from *f* draw $f\lambda$ parallel to CD cutting L*l*, M*m*, G*g*, in ϕ, π, and λ; and laſtly, draw the chords of the hyperbolic arcs *db*, *bk*, *ke*, *fl*, *lm*, *mg*.

Then in the firſt place, ſince CH : CD :: CL : CF, it will follow, *dividendo*, that CH : DH :: CL : FL, and *permutando* that CH : CL :: DH : FL. But (by *Simſon*'s *Con.* ſect. lib. 3. p. 16. *Corol.* 2.) we have CH : CL :: L*l* :: H*b*; therefore L*l* : H*b* :: DH : FL. conſequently (by *El.* 6. 14.) the parallelogram D*b* is equal to the parallelogram F*l*. In the ſame manner it may be ſhewn that the parallelogram H*k* is equal to the parallelogram L*m*, and K*e* to M*g*; conſequently the ſum of the three parallelograms

D*b*, H*k*, K*e*, is equal to the ſum of the three parallelograms F*l*, L*m*, M*g*; and theſe ſums, 'tis evident, would ſtill be equal if inſtead of three, as in the annexed figure, there were in each hyperbolic ſpace any number of parallelograms how great ſoever.

2*dly*, Since CH : CD :: CK : CH, and (by *Simſon*'s *Con.* ſect. lib. 3. p. 16. *Corol.* 2.) CH : CD :: D*d* : H*b*, and CK : CH :: H*b* : K*k*, it follows that D*d* : H*b* :: H*b* : K*k*; therefore, *dividendo*, N*d* : H*b* :: *bp* : K*k* and *permutando*, N*d* : *bp* :: H*b* : K*k*; therefore N*d* is greater than *bp*. And in the ſame manner it may be ſhewn that *bp*, and *a fortiori* that N*d*, is greater than *nk*. Therefore the parallelogram S*v* is greater than the parallelogram *n*P, and the parallelogram *tv* than the parallelogram *vp*; conſequently the parallelogram *d*R is greater than the ſum of the parallelograms N*t*, *vp*, *n*P, and therefore *a fortiori* than the ſum of the mixtilinear triangles N*bd*, *bpk*, *kne*, or than the difference of the ſum of the parallelograms D*b*, H*k*, K*e*, from the hyperbolic ſpace DE*ed*. And this, 'tis evident, will be true, whatever be the number of mixtilinear triangles contained between the points *d* and *e*.

3*dly*, 'Tis evident the number of intermediate proportionals CH, CK, between the lines CD, CE, and conſequently that of the correſpondent ordinates H*b*, K*k*, may be increaſed ſo far that the difference N*d* of the two contiguous ordinates D*d*, H*b*, ſhall be leſs than any aſſigned line whatſoever, and conſequently that the parallelogram *d*R ſhall be leſs than any aſſigned area whatſoever. Let this number therefore be ſo far increaſed, that the parallelogram *d*R ſhall be leſs than the rectangle Z: then 'tis plain, the ſum of the mixtilinear triangles contained between the points *d* and *e*, or the exceſs of the hyperbolic ſpace DE*ed* above the ſum of the parallelograms D*b*, H*k*, K*e*, inſcribed in it, will, *a fortiori*, be leſs than the rectangle Z, that is, than the exceſs of the hyperbolic ſpace DE*ed* above the hyperbolic ſpace FG*gf*. Therefore the ſum of the parallelograms D*b*, H*k*, K*e*, is greater than the hyperbolic ſpace FG*gf*, and *a fortiori*, greater than the ſum of the parallelograms F*l*, L*m*, M*g*, inſcribed in it. But it was before ſhewn that the ſum of the parallelograms D*b*, H*k*, K*e*, is always equal to the ſum of the parallelograms F*l*, L*m*, M*g*; therefore it is both equal to, and greater than, that ſum, which is impoſſible. Conſequently the hyperbolic ſpace DE*ed* is not greater than the hyperbolic ſpace FG*gf*. And in the ſame manner it may be ſhewn, that the hyperbolic ſpace FG*gf* is not greater than the hyperbolic ſpace DE*ed*. Therefore thoſe ſpaces are equal to each other. Q.E.D.

188. If the foregoing demonſtration be expreſſed in the conciſe language of infiniteſimals, or infinitely ſmall quantities, it will be as follows. Let each of the hyperbolic ſpaces be divided, by the inſertion of an equal but infinite number of mean proportionals between CD and CE and CF and CG and drawing the correſpondent ordinates parallel to CT, into an equal but infinite number of infinitely ſmall parallelograms; and it will appear from the known property of the aſymptotes, (to wit, that the ſeveral portions of one

of them taken from the center of the hyperbola are always reciprocally proportional to the correſponding ordinates drawn parallel to the other) that each parallelogram in the one hyperbolic ſpace is equal to the correſponding parallelogram in the other hyperbolic ſpace: conſequently the ſum of all the parallelograms in the former ſpace is equal to the ſum of all the parallelograms in the latter ſpace; that is, the whole former ſpace is equal to the whole latter ſpace. But I thought it beſt to give an accurate *ex abſurdo* demonſtration of this propoſition after the manner of the ancients, partly on account of its great uſe and importance, and partly that I might give an eaſy inſtance of the connection between the accurate method of demonſtration uſed by the antients in treating of the properties of curve-lined figures, and the conciſer method of infiniteſimals, or infinitely ſmall quantities, ſhewing how the former is virtually contained in the latter, and may, when occaſion requires, be ſubſtituted for it.

189. Having demonſtrated that, when the *ratios* of the ordinates are equal, the areas contained between them are equal likewiſe, it remains that we now prove that, when theſe hyperbolic ſpaces are unequal, they are proportional to the *ratios* of the ordinates that bound them.

Let VA*d* (*fig.* 71.) be an hyperbola whoſe vertex is A, center C, and aſymptotes CT, CD. On the aſymptote CD take the four lines CD, CH, CF, CL, of ſuch magnitudes that the *ratio* of the two former ſhall not be equal to the *ratio* of the two latter; and draw the ordinates D*d*, H*h*, F*f*, L*l*, parallel to the other aſymptote CT.

We are to prove that the hyperbolic ſpace FL*lf* is to the hyperbolic ſpace DH*bd* in the ſame proportion as the *ratio* of F*f* to L*l*, or of CL to CF, is to the *ratio* of D*d* to H*b*, or of CH to CD.

On the aſymptote CD take any number of lines in continued proportion to CF and CL, as CM and CG, and any other number of lines in continued proportion to CD and CH, as CK; and draw the correſpondent ordinates M*m*, G*g*, K*k*, parallel to the aſymptote CT. By *Art.* 187, the areas FL*lf*, LM*ml*, MG*gm* are all equal to each other; and likewiſe the areas DH*bd*, HK*kb* are equal to each other: conſequently the area FG*gf* is the ſame multiple of the area FL*lf* as the *ratio* of CG to CF is of the *ratio* of CL to CF, or as the *ratio* of F*f* to G*g* is of the *ratio* of F*f* to L*l*; and the area DK*kd* is the ſame multiple of the area DH*bd* as the *ratio* of CK to CD is of the *ratio* of CH to CD, or as the *ratio* of D*d* to K*k* is of the *ratio* of D*d* to H*b*. But ſince, by *Art.* 187, if the *ratio* of F*f* to G*g* is equal to the *ratio* of D*d* to K*k*, the area FG*gf* will be equal to the area DK*kd*, it follows that, if the *ratio* of F*f* to G*g* is greater or leſs than the *ratio* of D*d* to K*k*, the area FG*gf* will be accordingly greater or leſs than area DK*kd*. Therefore we have four quantities, to wit, the area FL*lf*, the area DH*bd*, the *ratio* of F*f* to L*l*, and the *ratio* of D*d* to H*b*; and of the firſt and third of thoſe quantities we have taken equimultiples at pleaſure, to wit, the area FG*gf*, and the *ratio* of F*f* to G*g*, and of the ſecond and fourth quantities we have taken other equimultiples at pleaſure, to wit, the area DK*kd*, and the *ratio* of D*d* to K*k*; and we have ſhewn that (whatever multiples theſe latter quantities are of the former) the equimultiple of the firſt will exceed, equal, or fall ſhort of, the equimultiple of the ſecond according as the equimultiple of the third exceeds, equals, or falls ſhort of, the equimultiple of the fourth. Conſequently theſe quantities are proportional. QED.

Of Hyperbolic Sectors.

190. From C the center of the hyperbola VA*d* (*fig.* 70.) draw the lines C*d*, C*e*, and let *b* be the point in which C*e* cuts D*d*. Then ſince, by the property of the aſymptote, D*d* is to E*e* as CE to CD, and the angles CD*d*, CE*e*, are equal, it follows (by *El.* 6, 15.) that the triangle CE*e* is equal to the triangle CD*d*; therefore, taking the triangle C*b*D from both ſides, the triangle C*bd* will be equal to the trapezium DE*eb*; conſequently, if we add on both ſides the curve-lined ſpace *dbe*, the hyperbolic ſector C*ed* will be equal to the hyperbolic trapezium DE*ed*.

191. In the ſame manner it may be ſhewn, that if we draw the lines C*f*, C*g*, (*fig.* 70.) the ſector C*gf* will be equal to the area FG*gf*. Therefore the ſector C*gf* is equal to the ſector C*ed*; that is, if from the center C, on the aſymptote CD, of an hyperbola VA*d* be taken four proportional lines CD, CE, CF, CG, ſo that the *ratio* of the two former lines CD, CE, ſhall be equal to the *ratio* of the two latter lines CF, CG, and the correſpondent ordinates D*d*, E*e*, F*f*, G*g*, be drawn parallel to the other aſyptote, and cutting the hyperbola in *d*, *e*, *f*, *g*, the hyperbolic ſector C*ed* belonging to the two former lines will be equal to the hyperbolic ſector C*gf* belonging

to the two latter lines; or the hyperbolic ſectors belonging to equal *ratios* of the ordinates, or portions of the aſymptote, are equal.

192. From *Art.* 123, 'tis evident, that if from the center C of the hyperbola VA*d* (*fig.* 71.) be drawn the lines C*d*, C*b*, C*f*, C*l*, the hyperbolic ſector C*bd* will be equal to the trapezium DH*bd*, and the ſector C*lf* to the trapezium FL*lf*. Therefore, by *Art.* 189, the ſector C*lf* is to the ſector C*bd* in the ſame proportion as the *ratio* of CL to CF, or of F*f* to L*l*, is to the *ratio* of CH to CD, or of D*d* to H*b*; that is, if, from the center C, on the aſymptote CD, of any hyberbola VA*d*, be taken any four lines CD, CH, CF, CL, that are not proportional, but are of ſuch magnitudes that the *ratio* of the two latter lines is not equal to the *ratio* of the two former, and the correſponding ordinates D*d*, H*b*, F*f*, L*l*, be drawn parallel to the other aſymptote CT, and meeting the hyperbola in *d*, *b*, *f*, *l*, the hyperbolic ſector C*lf* belonging to the two latter lines CL, CF, will bear the ſame proportion to the hyperbolic ſector C*bd* belonging to the two former lines CH, CD, as the *ratio* of the two latter lines, or of the correſponding ordinates F*f*, L*l*, bears to the *ratio* of the two former lines, or of the correſponding ordinates D*d*, H*b*; or the hyperbolic ſectors belonging to unequal *ratios* of the ordinates, or portions of the aſymptote, are proportional to thoſe *ratios*.

A remarkable analogy between circles and hyperbolas.

193. And hence we may obſerve a kind of analogy between the circle and hyperbola; to wit, that as the arcs and ſectors of a circle are natural and convenient meaſures of the magnitudes of angles, ſo the fore-mentioned trapeziums and ſectors of an hyperbola do moſt properly ſerve for meaſuring the magnitudes of *ratios*. And thus theſe figures mutually aſſiſt each other in the conſtruction of mathematical problems, it being extremely common, in problems that admit of ſeveral caſes, for ſome of the caſes to depend upon the meaſures of angles, and therefore to be conſtructible by means of circular arcs, or ſectors, and others of them to depend upon the meaſures of *ratios*, and conſequently to be conſtructible by means of the fore-mentioned hyperbolic ſectors and trapeziums.

Of the logarithmic ſpiral.

194. 'Tis evident, that any other kind of quantities as well as ſurfaces, may be made the meaſures, or logarithms, of *ratios*; of which we have the following inſtance in angles. Let ABDEFG (*fig.* 72.) be a curve of ſuch a nature, that, if from a certain fixed point C placed within it, the lines CA, CB, CD, CE, CF, CG, *&c.* be drawn at equal angles one from another and produced till they cut the curve in A, B, D, E, F, G, their *ratios* ſhall be equal one to another, or CA ſhall be to CB as CB to CD, and CB to CD as CD to CE, and ſo of the reſt. Then it will be

evident by repeating the demonſtration in *Art.* 189, that the angles made by the radii CA, CB, CE, *&c.* at the central point C, are meaſures, or logarithms, of the *ratios* of the radii that contain them; or, that, if we take any two unequal angles, as ACB and GCE, the angle GCE will be to the angle ACB in the ſame proportion as the *ratio* of the radii CG and CE that contain the former angle is to the *ratio* of the radii CB and CA that contain the latter angle. And hence this curve (which, it is evident, is a ſpiral, or goes round and round the center C in an infinite number of revolutions) is called the logarithmic ſpiral. It is alſo ſometimes called the æquiangular ſpiral, becauſe the angles CGF, CFE, CED, *&c.* made by the radii CG, CF, CE, *&c.* with the curve, or tangents to it in the points G, F, E, *&c.* are every where equal, as may eaſily be deduced from the property of the radii laid down in the foregoing definition.

195. From the foregoing article it follows, that if, with C as a center, (*fig.* 72.) and any line taken at pleaſure, as CH as a radius, a circle be deſcribed, the arches of this circle intercepted between the radii of the logarithmic ſpiral ABDEFG will be the logarithms of the *ratios* of thoſe radii of the ſpiral that contain the angles they ſubtend: for they are proportional to thoſe angles. Thus, if the circle cuts the radii CA, CB, CD, CE, CF, CG, *&c.* in the points *a*, *b*, *d*, *e*, *f*, *g*, &c. the circular arch *ge* will be to the circular arch *ab* in the ſame proportion as the *ratio* of the radii CG and CE is to the *ratio* of the radii CB and CA.

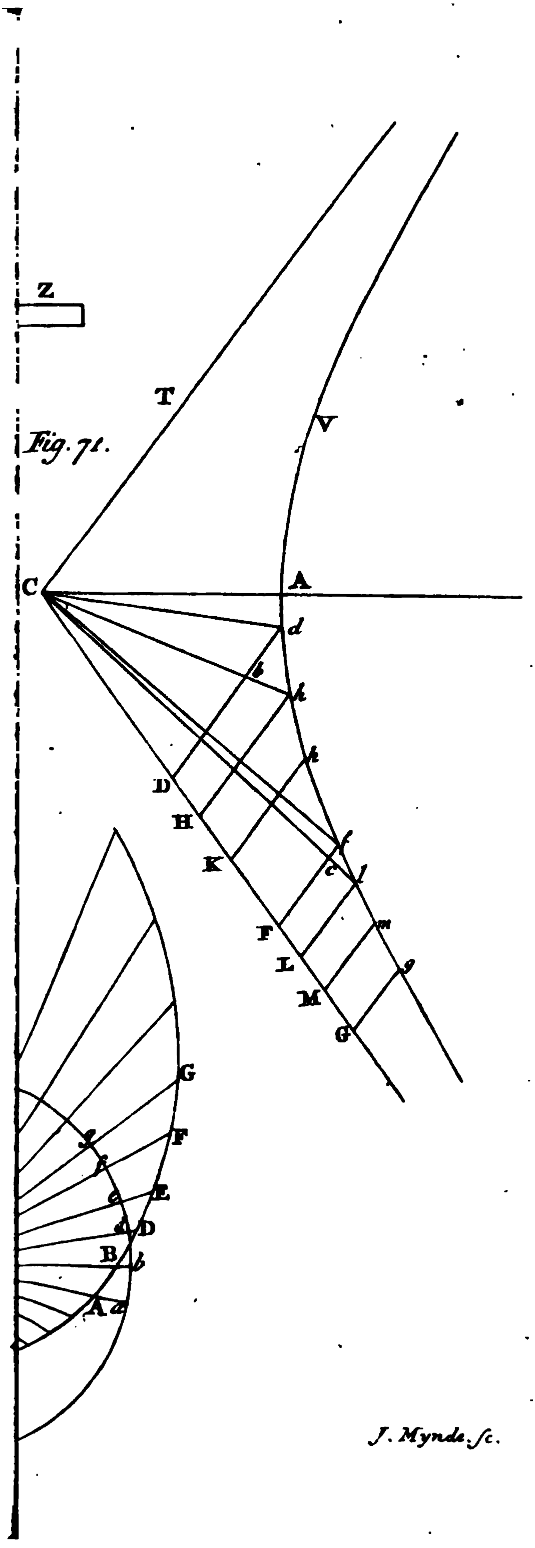
Of

Of the logarithmic curve.

196. From the foregoing articles it appears, that either ſurfaces, angles, or circular arcs, may be made meaſures, or logarithms, of *ratios:* but the ſimpleſt and moſt convenient method of exhibiting a ſyſtem of theſe meaſures ſeems to be by making uſe of right lines for that purpoſe by means of a logarithmic curve. Let any number of right lines, as OA, KC, PM, *&c.* (*fig.* 73.) that are in continued geometrical proportion, firſt to ſecond as ſecond to third, *&c.* be placed at right angles to the indefinite right line OP at equal diſtances from each other. The curve ACM, in which all the points A, C, M, *&c.* or the vertices of all the perpendicular ordinates OA, KC, PM, *&c.* are ſituated, is called a *logarithmic* curve. This curve may be continued on the other ſide of the point O by taking Oγ, $\gamma\pi$, *&c.* equal to each other, and to OK, KP, *&c.* and erecting at the points γ, π, *&c.* the lines γc, $\pi\mu$, *&c.* in continued proportion to KC and OA. And by biſecting the diſtances OK, KP, Oγ, $\gamma\pi$, *&c.* we may determine as many intermediate points in the curve as we pleaſe.

197. In this curve 'tis evident that the ordinates on the right hand of OA (*fig.* 73.) increaſe *ad infinitum*, or ſo as to become greater than any finite line whatſoever, and thoſe of the left-hand of OA decreaſe *ad infinitum*, or ſo as to become leſs than any aſſigned line whatſoever; but the curve and axis never actually meet: and hence the axis of a logarithmic curve is frequently called its aſymptote.

Fig. 71.



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198. The logarithmic curve is convex towards its aſymptote. Take any arc of the curve ACM (*fig.* 73.) as $A\mu$, and draw the chord of $A\mu$, and the ordinates AO, $\mu\pi$, perpendicular to the aſymptote, or axis, OP; and, biſecting the abſciſs $O\pi$ in γ, draw the ordinate γc cutting the arch $A\mu$ in c and its chord in q; and laſtly, draw the lines $\mu\beta$, $q\delta$, perpendicular to γq and OA, and meeting them in β, δ. Then it is evident that the ordinate γc will by the nature of the curve be a mean proportional between OA and $\pi\mu$; that is, OA : γc :: γc : $\pi\mu$; therefore *dividendo*, OA — γc : γc :: γc — $\pi\mu$: $\pi\mu$, and *permutando*, OA — γc : γc — $\pi\mu$ :: γc : $\mu\pi$; therefore the exceſs of OA above γc is greater than the exceſs of γc above $\pi\mu$; and conſequently, the ſum of theſe two exceſſes, or the exceſs of OA above $\pi\mu$, is greater than twice the exceſs of γc above $\pi\mu$. But becauſe the ſides of the triangle $q\mu\beta$ are reſpectively parallel to the ſides of the triangle $Aq\delta$, and the ſide $\mu\beta$ is equal to the ſide $q\delta$, it follows that theſe triangles are ſimilar and equal to each other: therefore $A\delta$ is $= q\beta$, and $A\delta + q\beta$, or the exceſs of OA above $\pi\mu$, is $= 2q\beta$, or twice the exceſs of γq above $\pi\mu$. Therefore twice the exceſs of γq above $\pi\mu$ is greater than twice the exceſs of γc above $\pi\mu$; conſequently γq is greater than γc, or the point c falls below the point q, that is, the arch $A\mu$ falls below its chord, or nearer than its chord to the axis OP, and conſequently is convex towards that axis. QED.

199. The abſciſſes of the axis of a logarithmic curve are meaſures, or logarithms, of the *ratios* of the ordinates that bound them. For if TK

and $\pi\gamma$ (*fig.* 73.) are any two unequal abſciſſes, and through the points T, K, π, γ, be drawn the ordinates TD, KC, γc, $\pi\mu$, it will appear by repeating the demonſtration in *Art.* 189, that the abſciſs TK is to the abſciſs $\pi\gamma$ in the ſame proportion as the *ratio* of the ordinates TD, KC, is to the *ratio* of the ordinates γc, $\pi\mu$, that is, that the abſciſſes TK and $\pi\gamma$ are logarithms of the *ratios* of TD to KC, and γc to $\pi\mu$. And hence it is that the curve ACM is called the logarithmic curve.

200. Let DL be a tangent to the curve ACM (*fig.* 73.) in the point D, meeting the axis OP in L. Draw the ordinate *dt* infinitely near to DT, and from *d* draw *db* perpendicular to DT. 'Tis evident, the infinitely ſmall triangle *db*D will be ſimilar to the triangle DLT; therefore D*b* : *db*, or *t* T, :: DT : LT; that is, if the curve be conceived to be generated by the motion of the ordinates DT moving parallel to itſelf along the axis OP, and at the ſame time increaſing in length in ſuch a manner that its upper extremity D ſhall trace out the curve ACM, the naſcent increment of the ordinate will be to the contemporary increment of the abſciſs at any propoſed inſtant of time, as for inſtance, when the ordinate is in the ſituation DT, as the magnitude DT of the ordinate at that inſtant of time is to the correſponding ſubtangent LT; or, the limit of the *ratio* of the contemporary increments of the ordinate and abſciſs is equal to the *ratio* of the ordinate to the ſubtangent.

201. Hence it is evident, that the velocity with which the ordinate increaſes, is to the velocity with which the abſciſs increaſes, or to

the velocity of the parallel motion of the ordinate, as the ordinate to the ſubtangent; or, in other words, the fluxion of the ordinate is to the fluxion of the abſciſs as the ordinate to the ſubtangent: and this, whether the velocity of the parallel motion of the ordinate, or the fluxion of the abſciſs, be ſuppoſed to be uniform or variable.

202. The ſubtangents belonging to different points of the ſame logarithmic curve are all of the ſame magnitude; or, if we take any two points at pleaſure, as D and A, in the logarithmic curve ACM (*fig.* 73.) and draw the tangents DP, AF meeting the axis in L and P, and the ordinates DT, AO, meeting it in T and O, the ſubtangent FO will be equal to the ſubtangent PT. For let dt be an ordinate infinitely near to DT; take $OH = tT$, and draw the ordinate Ha; and from d and a draw db and ab perpendicular to DT and AO. Then from the nature of the curve 'tis evident that $AO : Ha :: DT : dt$; whence, *dividendo*, we have $Ab : AO :: Db : DT$, and *permutando*, $Ab : Db :: AO : DT$. But becauſe the triangle Aab is ſimilar to the triangle AFO, and the triangle Ddb to the triangle DPT, it follows, that $ab : Ab :: FO : AO$, and $Db : db :: DT : PT$. We have therefore the three following proportions; $ab : Ab :: FO : AO$; $Ab : Db :: AO : DT$; and $Db : db :: DT : PT$. Therefore, *ex æquo*, $ab : db :: FO : LT$, and conſequently, becauſe $ab = db$, FO is $=$ PT. QED.

203. Hence it follows, that the ordinates drawn at the extremities of the ſubtangent of a logarithmic curve are always in the ſame propor-

tion to each other, at what point ſoever of the curve the ſubtangent be taken; or, the ſubtangent of a logarithmic curve is the meaſure, or logarithm, of a given *ratio*. Thus, the proportion of the ordinates DT, PM, is the ſame with that of the ordinates AO, F*f*. This proportion is found by calculation to be that of 2.718,281,828,459, &c. to 1, or of 1 to .367,879, 441,171, &c.

204. And hence we have a method of deſcribing a logarithmic curve by finding as many points in it as we pleaſe, if its ſubtangent only be given. For if the magnitude of PT is known, draw the ordinates DT, PM, perpendicular to it in the points T, P, and take DT to PM as 2.718, &c. to 1. This done, take any number of continual proportionals to L*l* and DT, making firſt L*l*, and then DT, the firſt term of the proportion, and place theſe proportionals on each ſide of the ſubtangent LT at right angles to it, and at diſtances equal to LT one from another. 'Tis evident the vertexes of all theſe proportionals will be ſituated in a logarithmic curve whoſe ſubtangent is LT. If the intermediate points of the curve that lie between theſe proportionals be required, they may be found by continually biſecting the abſciſſes of the axis intercepted between them, and erecting mean proportionals between them at the ſeveral points of biſection.

205. Another way of performing the ſame thing is this that follows. In the indefinite right line OP (*fig.* 73.) take LT equal to the given ſubtangent of the curve that is to be deſcribed, and at the point O draw the right line TD perpen-

pendicular to OP; and of any magnitude whatſoever; and let this line be moved parallel to itſelf on the line OP firſt towards the right-hand, and then towards the left-hand ſide of the figure with any velocity whatſoever, uniform or variable: and, as it is moving towards the right-hand ſide of the figure, let it be ſuppoſed to increaſe in length at ſuch a rate that the velocity of its increaſe ſhall be to the velocity of its parallel motion at every inſtant of time during its increaſe, or the naſcent increment of DT to the contemporary increment of OT, as the magnitude of DT at the ſame inſtant of time to the given line LT; and in like manner, as it is moving towards the left-hand ſide of the figure, let it be ſuppoſed to decreaſe in length at ſuch a rate that the velocity of its decreaſe ſhall be to the velocity of its parallel motion at every inſtant of time during its decreaſe as the magnitude of DT at the ſame inſtant to the given line LT. 'Tis evident, that by theſe two motions there will be deſcribed the logarithmic curve μc ACM whoſe ſubtangent is equal to the given line LT.

206. From the laſt articles it follows, that in different logarithmic curves, or ſuch wherein the logarithms of the ſame *ratio*, or the abſciſſes of the axis intercepted between terms that are in the ſame proportion to each other, are unequal, (for this is evidently the only difference there can be between one logarithmic curve and another) theſe unequal logarithms of the ſame *ratio* are proportional to the ſubtangents of the curves, and the curves are ſimilar to each other. For whenever a curve is determined by one line only, or is given when one line is given, its ſpecies muſt

be given, or by varying that determining line, the curve can only change its magnitude without any change in its form or ſpecies, or in the proportion of the lines that correſpond to each other, or are determined by like conſtructions. Thus all circles, as they are determined by their diameters, are ſimilar to each other; all parabolas, as they are determined by their parameters, or are given when their parameters are given, are ſimilar to each other; all equilateral, or rectangular hyperbolas, as they are determined by their tranverſe axes, are ſimilar to each other; and all cycloids, as they are determined by the diameters of their generating circles, are alſo ſimilar to each other. But as theſe properties of logarithmic curves are of ſome importance, and will tend much to forming a juſt conception of them, we ſhall here demonſtrate them in a more exact and formal manner, as follows.

207. The logarithms of a given *ratio* taken on the axis of two different logarithmic curves are in the ſame proportion to each other as the ſubtangents of thoſe curves. Let OA and γc be any two ordinates in the logarithmic curve ACM (*fig.* 73.) AF a tangent to the curve in A, and FO its ſubtangent; and in the curve $\alpha\chi nr$ (*fig.* 74.) let Rr and Nn be any two ordinates that are in the ſame proportion to each other as OA and γc, $r\phi$ a tangent to the curve $\alpha\chi nr$ in r, and $R\phi$ its ſubtangent. We are to ſhew that the abſciſs RN is to the abſciſs $O\gamma$ in the ſame proportion as the ſubtangent $R\phi$ to the ſubtangent FO. In *fig.* 73, draw the ordinate Ha infinitely near to OA, and in *fig.* 74, take the ordinate

Q*q* that is in the ſame proportion to R*r* as H*a* is to OA; and from the points *a*, *q*, draw the lines *ab*, *q*δ, perpendicular to OA and R*r* and meeting them in *b*, δ. Then ſince R*r* : Q*q* :: OA : H*a*, it follows, *dividendo*, that δ*r* : R*r* :: A*b* : OA, and, *permutando*, δ*r* : A*b* :: R*r* : OA. Further, from the ſimilarity of the triangles AOF and A*ba* it follows, that AO : FO :: A*b* : *ab*; and from the ſimilarity of the triangles R*r*φ and *r*δ*q*, it follows, that Rφ : R*r* :: *q*δ : δ*r*. We have therefore the three following proportions; *q*δ, or QR : δ*r* :: Rφ : R*r*; δ*r* : A*b* :: R*n* : AO; and A*b* : *ab*, or OH, :: AO : FO. Therefore *ex æquo* we have QR : OH :: Rφ : FO; that is, RQ the logarithm of the infinitely ſmall *ratio* of R*r* to Q*q* in the curve *a*χ*nr* is to OH the logarithm of the ſame *ratio* in the curve ACM as Rφ the ſubtangent of the former curve is to FO the ſubtangent of the latter curve. But, from the nature and definition of a logarithmic curve, 'tis evident that the logarithms of any other *ratio*, taken in the curves *a*χ*nr* and ACM, as for inſtance, the logarithms NR and Oγ of the *ratio* of R*r* to N*n*, or AO to γ*c*, are to each other in the ſame proportion as the logarithms RQ and OH of the infinitely ſmall *ratio* of R*r* to Q*q*, or OA to H*a*, taken in the ſame curves; (for the former logarithms may be conſidered as equimultiples of the latter;) therefore the logarithms RN and Oγ of the equal *ratios* of R*r* to N*n*, and AO to γ*c*, taken in the logarithmic curves *a*χ*nr* and ACM are to each other in the ſame proportion as Rφ and FO the ſubtangent of thoſe curves. QED.

208. It eaſily appears, that if the logarithms of a given *ratio* in two different logarithmic curves, or the ſubtangents of thoſe curves, are equal, the curves will be exactly equal and ſimilar in every reſpect, or, if placed one upon another, will exactly coincide. For if the *ratio* of OA to γc (in *fig.* 73.) be equal to that of Rr to Nn (in *fig.* 74.) ; and Nn is equal to γc, Rr will be equal to OA : if therefore the logarithm RN is equal to the logarithm Oγ, and Nn is placed upon γc ſo as to coincide with it, Rr will coincide with OA, and every other ordinate in the curve $\alpha\chi nr$ will coincide with the ordinate equal to it in the curve ACM, ſo that the former of theſe two figures will exactly cover the latter, or they will be both in every reſpect equal and ſimilar. The ſame will be evident if Nn is not equal to γc; for we may then take in ſome other part of the figure $\alpha\chi nr$ the ordinate $\omega\alpha$ equal to OA, and $\Gamma\Delta = \gamma c$; and if the logarithm Oγ is equal to the logarithm $\omega\Gamma$, and $\Gamma\Delta$ be placed upon γc, the ordinate $\omega\alpha$ will coincide with OA, and every other ordinate in the curve $\alpha\chi nr$ will coincide with the ordinate equal to it in the curve ACM, ſo that the former of theſe two figures will exactly cover the latter, or be exactly equal and ſimilar to it, as before.

209. If the logarithms of a given *ratio* in two different logarithmic curves, or the ſubtangents of thoſe curves, are unequal, the whole figures, though not equal and coincident as before, will nevertheleſs be ſimilar to each other; that is, whatever finite portion of a given logarithmic curve is propoſed, we may always find a portion in any other logarithmic curve of a dif-

ferent ſubtangent, that is ſimilar to the former portion; or whoſe parts are all greater or leſs than the correſpondent parts (or parts determined in the ſame manner, or by the ſame conſtructions) of the former portion in the ſame given *ratio*, or which may be derived from the former portion by increaſing or diminiſhing all the lines that bound and determine it in a given *ratio*. Let AC be any portion of the logarithmic curve ACM (*fig.* 73.) whoſe axis is OP, ſubtangent FO, and the ordinates at A and C, OA and KC; and let $\alpha\chi nr$ (*fig.* 74.) be any other logarithmic curve whoſe ſubtangent is greater than FO. Draw the axis ωR of this curve, and in it find the point R at which the ordinate R*r* is to KC as the ſubtangent Rϕ of the curve $\alpha\chi nr$ is to FO the ſubtangent of ACM; and let N*n* be another ordinate in the curve $\alpha\chi nr$ that is to R*r*, as OA to KC. Then we ſhall have, by *Art.* 207. NR : OK :: Rϕ : FO :: R*r* : KC :: N*n* : OA; that is, the ordinates R*r* and N*n* and the abſciſs NR, are all of them greater than the correſpondent ordinates KC and OA, and the abſciſs OK, in the ſame given *ratio* of the ſubtangents Rϕ and FO. And, if NR be divided into any number of equal parts, and OK be likewiſe divided into the ſame number of equal parts, and ordinates be drawn in both figures at the ſeveral points of diviſion in theſe lines, the ſame may be ſhewn concerning theſe intermediate ordinates between N*n* and R*r*, and OA and KC; to wit, that all thoſe in the figure NR*rn* are greater than the correſpondent ordinates in the figure OKCA in the ſame given proportion creaſing of the ſubtangents. Therefore by increaſing the ordinates and the baſe OK of the figure OKCA in the given *ratio* of the ſubtangent Rϕ to the ſubtangent FO, it will become equal to, and exactly coincide with the figure NR*rn*: conſequently the figure NR*rn* is ſimilar to the figure OKCA. QED.

Note, The having all their dimenſions either greater or leſs the one than the other in the ſame proportion, is ſo conſtant a concomitant of the idea of the ſimilarity of two figures that I have in the courſe of this article conſidered it as the very eſſence and definition of ſimilarity: for if ſimilarity be defined in a different manner; as, for inſtance, if two curve-lined figures are ſaid to be ſimilar, when, whatever right-lined figure be inſcribed in, or circumſcribed about, the one, it is poſſible to inſcribe, or circumſcribe, a right-lined figure ſimilar to the former, in or about the other, (which is the definition of the ſimilarity of curve-lined figures given by the marquis *de l'Hoſpital*, in the 5th book of his excellent treatiſe on the conic ſections;) 'tis evident, that where the former property of having all the dimenſions of the one figure greater or leſs than thoſe of the other in the ſame proportion is found, this latter property will be found likewiſe, or the figures will be ſimilar in this latter ſenſe.

210. From hence it follows, that the curve-line *nr* is to the curve-line AC as the ſubtangent Rϕ to the ſubtangent FO. For let us ſuppoſe the intermediate ordinates inſerted between N*n* and R*r*, and between OA and KC, to be infinite in

number; and 'tis evident the quadrilateral figures into which the figures NR*rn* and OKCA will be thereby divided, whoſe upper ſides are the ſeveral portions of the curve lines *nr* and AC, will in this caſe become rectilinear trapeziums, becauſe the portions of the curves, being infinitely ſmall, will become right lines, and coincide with their chords: and of theſe trapeziums thoſe into which the figure NR*rn* is divided will be reſpectively ſimilar to thoſe into which the figure OKCA is divided: conſequently the upper, or oblique, ſide of every trapezium in the figure NR*rn* will be to the upper, or oblique ſide of the correſponding trapezium in the figure OKCA, in the ſame *ratio* as their baſes, or as the ſubtangent Rφ to the ſubtangent FO; that is, every infinitely ſmall part of the curve line *nr* is to the correſponding part of the curve line AC as the ſubtangent Rφ to the ſubtangent FO. Therefore the whole curve line *nr* is to the whole curve line AC as Rφ to FO. QED.

211. The area of the figure NR*rn* is to the area of the figure OKCA as the ſquare of the ſubtangent Rφ to the ſquare of the ſubtangent FO, For each of the infinitely ſmall trapeziums into which the figure NR*rn* is divided is to the correſponding trapezium in the figure OKCA in the duplicate *ratio* of their homologous ſides, or of the ſubtangents Rφ and FO: therefore the whole area NR*rn* is to the whole area OKCA in the duplicate *ratio* of Rφ to FO, or as Rφ*q* to FO*q*, QED.

Of the analogy between the hyperbola and the logarithmic curve.

212. The parallelogram of an hyperbola

(which, by the property of the aſymptotes, is every where of the ſame magnitude) is the logarithm, or meaſure, of the ſame *ratio* in the hyperbola, of which the ſubtangent of a logarithmic curve is the meaſure in that curve; or, the ordinates which bound that curvilinear area of an hyperbola that is equal to its parallelogram are to each other in the ſame proportion as the ordinates to the axis of a logarithmic curve that are drawn through the extremities of an abſciſs equal to its ſubtangent. For from any point *b* of the hyperbola VA*d* (*fig.* 70.) draw the ordinates *bu*, *b*H, parallel to the aſymptotes CD, CT; and, taking the point D infinitely near to H, draw the ordinate D*d*: and in the logarithmic curve ACM (*fig.* 73.) take the ordinate H*a* = CD, and OA = CH: and draw the tangent AF, and the line *ab* parallel to the axis. Then 'tis evident we ſhall have DH = A*b*, and conſequently DH : CH :: A*b* : OA. But the curvelined area DH*bd*, being infinitely ſmall, is equal to the parallelogram D*b*, and therefore is to the parallelogram of the hyperbola CH*bu* (*El.* 6. 1.) as DH to CH: and, becauſe the triangles A*ba*, AOF are ſimilar, *ab* is to FO as A*b* to OA. Therefore D*h* : C*b* :: *ab*, or OH, : FO; that is, the infinitely ſmall area, or parallelogram, D*b* is to the parallelogram C*b* of the hyperbola, as the infinitely ſmall abſciſs OH to the ſubtangent FO. Conſequently the *ratio* of D*d* to H*b*, or of CH to CD, is to the *ratio* of the ordinates that terminate the hyperbolic area that is equal to the parallelogram C*b*, as the *ratio* of OA to H*a* to the *ratio* of the ordinates OA, F*f*, drawn through the extremities of the ſubtangent FO. But the *ratio* of OA to H*a* is equal to the *ratio* of CH to CD; therefore the *ratio* of the ordinates that terminate the hyperbolic area equal to the parallelogram C*b*, is equal to the *ratio* of the ordinates OA, F*f*, drawn thro' the extremities of the ſubtangent FO. QED.

Scholium.

We have been obliged in the foregoing demonſtration to compare together infinitely ſmall and finite quantities, to wit, the infinitely ſmall area, or parallelogram, D*b* with the parallelogram C*b*, and the infinitely ſmall abſciſs OH with the ſubtangent FO, becauſe the two infinitely ſmall quantities D*b* and OH being of different kinds could not be compared together. Now as this is a compariſon which has not yet been made in any of the foregoing articles of this propoſition, and which cannot be explained by means of the limits of the *ratios* in the ſame manner as the proportion of two infinitely ſmall quantities that vaniſh in a finite *ratio* one to the other, I ſhall here endeavour to ſhew what is to be underſtood by making theſe compariſons, and afterwards repeat the foregoing demonſtration in the accurate language of finite quantities.

In the firſt place then, when it is concluded from the ſimilarity of the triangles A*ba*, AOF (*fig.* 73.) that A*b* : AO :: OH : FO, the meaning of this aſſertion is this. If the difference A*b* of the two ordinates OA, H*a* is very ſmall in compariſon of either of thoſe ordinates themſelves, the *ratio* of A*b* to AO will be nearly equal to the *ratio* of the abſciſs OH to the ſubtangent FO; and this difference A*b* may be taken ſo ſmall in compariſon to either of the ordinates AO, H*a*, that the *ratio* of A*b* to AO ſhall differ leſs from the *ratio* of OH to FO, than by any aſſigned *ratio* how ſmall ſoever.

2*dly*, When it is concluded from the infinite ſmallneſs of the area DH*bd*, and its equality with the parallelogram D*h*, that its proportion to the parallelogram C*b* is the ſame with that of DH to CH, the meaning of this aſſertion is this: if the area DH*bd* be very ſmall in compariſon of the parallelogram C*b*, its proportion to that parallelogram will be nearly equal to the proportion of DH to CH; and this area may be taken ſo ſmall in compariſon of C*b*, or the point H may be taken ſo near to the point D, that the proportion of the area DH*bd* to the parallelogram C*b* ſhall differ leſs from the proportion of DH to CH than any aſſigned *ratio* how ſmall ſoever.

3*dly*, When 'tis concluded from, the ultimate equality of the *ratios* of, DH to CH, and the area DH*bd* to the parallelogram C*b*, the ultimate equality of the *ratios* of, A*b* to AO, and OH to FO, and the conſtant equality of the *ratios* of, A*b* to AO, and DH to CH, that DH*bd* : C*b* :: OH : FO, the meaning of this aſſertion is this: if the difference A*b* of the ordinates AO, H*a*, or the difference DH of the lines CH, CD, which are reſpectively equal to AO, H*a*, be very ſmall in compariſon of AO and H*a*, the *ratio* of the little hyperbolic area DH*bd* to the parallelogram C*b* will be nearly equal to the *ratio* of the ſmall abſciſs OH to the ſubtangent FO; and the difference A*b* may be taken ſo ſmall in compariſon of OA, and Ha, that the *ratio* of DHbd to Cb ſhall differ leſs from the *ratio* of OH to FO than by any aſſigned *ratio* how ſmall ſoever.

4thly, We may obſerve further, that as long as Ab is of any finite magnitude, how ſmall ſoever, the tangent AF will paſs without the curve Aa, and conſequently will not cut the line ab in the point a, but in ſome other point, which call e, that is nearer than a to the point b, ſo that the line eb, will be leſs than ab. Now the triangles Aeb, AOF, are ſtrictly ſimilar, even tho' Ab be finite; therefore Ab : AO :: eb : FO; conſequently Ab is to AO as ab, or OH, to a line greater than FO, which call x. But Ab may be taken ſo ſmall in compariſon of AO, that the *ratio* of ab to eb ſhall approach as nearly as we pleaſe to a *ratio* of equality; therefore Ab may be taken ſo ſmall in compariſon of AO that the *ratio* of x to FO ſhall approach as nearly as we pleaſe to a *ratio* of equality, or that x ſhall exceed FO by as ſmall a quantity as we pleaſe; that is, the ſubtangent FO is the limit of the magnitude of x, or of a fourth proportional to the ſmall, but finite, difference Ab of the ordinates OA, Ha, the ordinate OA, and the correſpondent ſmall abſciſs, or logarithm, OH.

5thly, In like manner in the hyperbola VAd (*fig.* 70.) the difference DH, when finite, is to the line CH accurately as the parallelogram Db to the parallelogram Cb, and conſequently as the area DHbd to a ſpace greater than Cb, which call yy: but DH may be taken ſo ſmall in compariſon of CH that the *ratio* of DHbd to Db ſhall approach as near as we pleaſe to a *ratio* of equality: therefore DH may be taken ſo ſmall, in compariſon of CH, that the *ratio* of *yy* to C*b* ſhall approach as near as we pleaſe to a *ratio* of equality, or that *yy* ſhall exceed C*b* by as ſmall a quantity as we pleaſe; that is, the parallelogram C*b* is the limit of the magnitude of *yy*, or of a fourth proportional to DH the ſmall, but finite, difference of the lines CH, CD, the line CH, and the correſpondent ſmall area, or logarithm, DH*bd*.

Theſe things being premiſed, the propoſition may be demonſtrated as follows: DH*bd* : *yy* :: DH : CH :: A*b* : OA :: OH : *x*; therefore the *ratio* whereof *yy* is the logarithm in the hyperbola VA*d* (*fig.* 70.) is to the *ratio* whereof DH*bd* is the logarithm in the ſame hyperbola; or to the *ratio* of CH to CD, as the *ratio* whereof *x* is the logarithm in the logarithmic curve ACM (*fig.* 73.) is to the *ratio* whereof OH is the logarithm in the ſame curve, or to the *ratio* of OA to H*a*: but the *ratio* of OA to H*a* is equal to the *ratio* of CH to CD: therefore the *ratio* whereof *yy* is the logarithm in the hyperbola VA*d* is equal to the *ratio* whereof *x* is the logarithm in the logarithmic curve ACM. And as this is true in all the ſtates of *yy* and *x*, it muſt be true likewiſe in their laſt ſtate when *yy* = C*b*, and *x* = FO; that is, the *ratio* whereof the parallelogram C*b* is the logarithm in the hyperbola VA*d* muſt be equal to the *ratio* whereof the ſubtangent FO is the logarithm in the logarithmic curve ACM. But if this be denied, let either of them, as ſuppoſe the parallelogram C*b*, be the logarithm of the greater *ratio*. Then, ſince by diminiſhing A*b* we may diminiſh *x* till it exceeds FO by leſs than any aſſigned quantity, and conſequently till the *ratio* whereof it is the logarithm exceeds the *ratio* whereof FO is the logarithm by leſs than any aſſigned *ratio* how ſmall ſoever, let x be ſo far diminiſhed that the exceſs of the *ratio* whereof it is the logarithm above the *ratio* whereof FO is the logarithm ſhall be leſs than the exceſs of the *ratio* whereof C*b* is the logarithm above the *ratio* whereof FO is the logarithm; and it is evident that the *ratio* whereof x is the logarithm will be leſs than the *ratio* whereof C*b* is the logarithm: therefore *a fortiori* the *ratio* whereof x is the logarithm will be leſs than the *ratio* whereof yy, which is greater than C*b*, is the logarithm. But it was before ſhewn that the *ratio* whereof x is the logarithm is always equal to the *ratio* whereof yy is the logarithm: therefore it is both equal and leſs than that *ratio* at the ſame time; which is impoſſible. Therefore the *ratio* whereof C*b* is the logarithm is not greater than the *ratio* whereof FO is the logarithm. In the ſame manner it may be ſhewn that the *ratio* whereof FO is the logarithm is not greater than the *ratio* whereof C*b* is the logarithm: conſequently theſe *ratios* are equal: QED.

213. The logarithms of equal *ratios* in different hyperbolas (whether thoſe hyperbolas be ſimilar or diſſimilar, that is, whether the proportion of their tranſverſe and ſecond axes be the ſame or different) are to each other in the ſame proportion as the parallelograms of thoſe hyperbolas. For the logarithm of any one given *ratio* in any one hyperbola is to the logarithm of any

other given *ratio* in the ſame hyperbola as the former *ratio* is to the latter, and therefore as the logarithm of the former *ratio* in any other hyperbola to the logarithm of the latter *ratio* in that other hyperbola; and *permutando*, the logarithm of the former *ratio* in the former hyperbola is to its logarithm in the latter hyperbola as the logarithm of the latter *ratio* in the former hyperbola is to its logarithm in the latter hyperbola: therefore if the latter *ratio* be taken equal to that *ratio* whereof the parallelogram of the hyperbola is the logarithm, it will be evident that the logarithm of any given *ratio* in one hyperbola is to the logarithm of the ſame *ratio* in any other hyperbola as the parallelogram of the former hyperbola to to the parallelogram of the latter hyperbola. QED.

214. In the ſame manner we may demonſtrate what has been already ſhewn in *Art.* 207, concerning logarithmic curves, to wit, that the logarithms of equal *ratios* in two different logarithmic curves are to each other in the ſame proportion as the ſubtangents of thoſe curves. For the logarithm of any one given *ratio* in one logarithmic curve is to the logarithm of any other given *ratio* in the ſame logarithmic curve (as the former *ratio* is to the latter, and therefore) as the logarithm of the former *ratio* in any other logarithmic curve to the logarithm of the latter *ratio* in that other logarithmic curve; and *permutando*, the logarithm of the former *ratio* in the former logarithmic curve is to its logarithm in the latter logarithmic curve as the logarithm of the latter *ratio* in the former logarithmic curve is to its logarithm in the latter logarithmic curve: there-

fore if the latter *ratio* be taken equal to that whereof the ſubtangent is the logarithm, it will be evident that the logarithm of any given *ratio* in one logarithmic curve is to the logarithm of the ſame *ratio* in any other logarithmic curve as the ſubtangent of the former curve is to the ſubtangent of the latter curve. QED.

215. From *Art.* 213, it is evident that the quadratures of hyperbolas of all ſpecies may be reduced to the quadrature of an hyperbola of any one given ſpecies, as, for inſtance, an equilateral hyperbola; or, that if, when the tranſverſe and ſecond axis of any one hyperbola are given, we are able to determine the magnitude of the curvilinear area that is intercepted between two ordinates that are to each other in any known, or given *ratio*, we ſhall likewiſe be able to determine in any other hyperbola whereof the tranſverſe and ſecond axes are given, the magnitude of the curvilinear area that is intercepted between two ordinates that are in the ſame proportion to each other as the two former ordinates. For we need only ſay, as the parallelogram of the former hyperbola is to the parallelogram of the latter hyperbola ſo is the area of the former hyperbola to the coreſpondent area (or area that is bounded by ordinates in the ſame proportion) of the latter hyperbola. QED.

216. Since the parallelogram and aſymptotic areas of an hyperbola are to each other in the ſame proportions as the ſubtangent and abſciſſes of the axis of a logarithmic curve correſponding to the ſame *ratios*, it follows that they will be equal to the rectangles whoſe baſes are the ſubtangent and abſciſſes of the axis of the logarith-

mic curve, and whoſe height is one and the ſame given right line, to wit, the altitude of a rectangle whoſe area is equal to that of the parallelogram of the hyperbola, and whoſe baſe is the ſubtangent of the logarithmic curve.

Recapitulation of the foregoing properties of the logarithmic curve.

217. We have now gone through all thoſe properties of the logarithmic curve that have any relation to the preſent ſubject, or may tend to give a clear notion of its figure and deſcription; which (as for the ſake of avoiding all kind of obſcurity and confuſion on this not very eaſy ſubject, we have demonſtrated them at great length) it may not be amiſs to recapitulate in a few words as follows. After giving a definition of this curve, we have ſhewn, in the firſt place, that its axis is an aſymptote to it; 2*dly*, that the curve is convex towards its axis, or aſymptote; 3*dly*, that the abſciſſes of the axis are logarithms of the *ratios* of the ordinates that bound them; 4*thly*, that the ſubtangent of this curve is every where of the ſame magnitude, and is therefore every where the logarithm of the ſame given *ratio*, to wit (as is found by calculation) the *ratio* of 2.718,281,828,459, &c. to 1; 5*thly*, that when the ſubtangent is given, the curve may be deſcribed in two different manners; the one by erecting perpendicularly to the axis at proper diſtances from each other a variety of mean and continual proportionals between, and to, the ordinates drawn through the extremities of the ſubtangent, which ordinates muſt be taken to each in the foreſaid proportion of 2.718, &c. to 1;

the other by a continued motion, to wit, by moving a right line upon the axis, with any velocity whatſoever, and ſuppoſing it to increaſe during its motion at ſuch a rate that the velocity of its increaſe ſhall always be to the velocity of its parallel motion, or the naſcent increment of the ordinate to the contemporary increment of the abſciſs, at every inſtant of time during its generation, as the magnitude of the ordinate at the ſame inſtant of time to the given line which is equal to the ſubtangent of the curve to be deſcribed; 6*thly*, that the logarithms of equal *ratios* in different logarithmic curves are proportional to the ſubtangents of thoſe curves; of which we have given two demonſtrations, the one in *Art.* 207, the other in *Art.* 214; and we have ſhewn that the ſame thing holds in hyperbolas, or that the logarithms of equal *ratios* in different hyperbolas are proportional to the parallelograms of thoſe hyperbolas; 7*thly*, that all logarithmic curves are ſimilar, and the correſpondent lines in them proportional to their ſubtangents, and conſequently the correſpondent areas proportional to the ſquares of their ſubtangents; 8*thly*, that the *ratio* whereof the parallelogram of any hyperbola is the logarithm in that hyperbola is equal to the *ratio* whereof the ſubtangent of any logarithmic curve is the logarithm in that curve; and therefore 9*thly*, that the parallelogram and areas of any hyperbola are equal to a ſet of rectangles whoſe baſes are the ſubtangent and correſpondent abſciſſes of the axis of a logarithmic curve, and height the altitude of a rectangle whoſe baſe is the ſubtangent of the logarithmic curve, and area equal to the parallelogram of the hyperbola;

10*thly*,

10*thly*, that the ſubtangent of a logarithmic curve is a fourth proportional to the infinitely ſmall difference of any two contiguous ordinates, the greater, or either, of thoſe ordinates, and the correſpondent infinitely ſmall logarithm, or the abſciſs of the axis intercepted between thoſe ordinates; or, to ſpeak more accurately, the ſubtangent of a logarithmic curve is the limit of the magnitude of a fourth proportional to the ſmall, but finite, difference of any two contiguous ordinates, the greater, or either, of thoſe ordinates, and the correſpondent logarithm, or the ſmall abſciſs of the axis intercepted between thoſe ordinates: and we have ſhewn likewiſe that the ſame thing holds in hyperboles, or that the parallelogram of an hyperbola is a fourth proportional to the infinitely ſmall difference of any two portions of the aſymptote taken from the center of the hyperbola, the greater, or either, of thoſe lines, and the correſpondent infinitely ſmall logarithm, or the aſymptotic area whoſe baſe is the foreſaid difference; or, to ſpeak more accurately, that the parallelogram of an hyperbola is the limit of the magnitude of a fourth proportional to the ſmall, but finite, difference of any two portions of the aſymptote taken from the center of the hyperbola, the greater, or either, of thoſe portions, and the correſpondent ſmall logarithm, or the aſymptotic area whoſe baſe is the foreſaid ſmall difference.

Pl

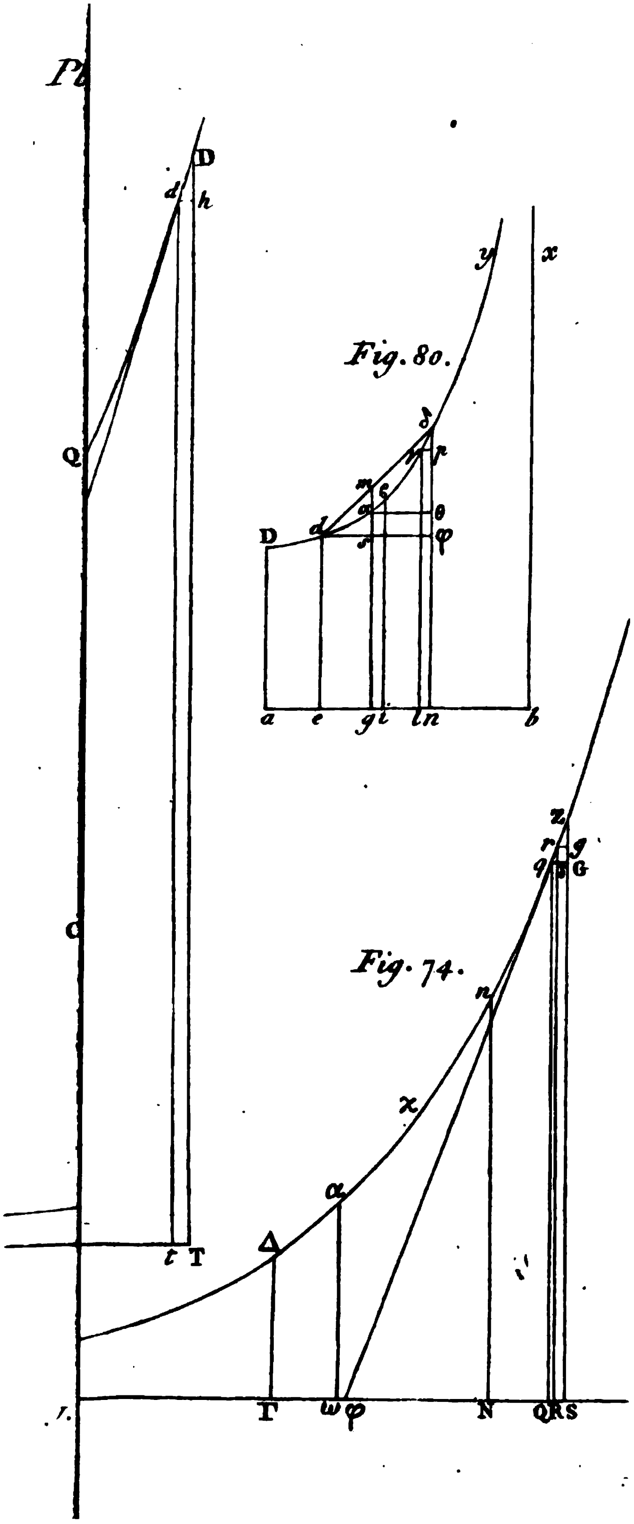


Fig. 80.

Fig. 74.

The plan of the remaining part of this diſſertation.

218. Having in the preceding part of this *Corollary* explained firſt in what ſenſe *ratios* may be ſaid to bear proportions one to another, and then defined logarithms to be any meaſures of *ratios* of what kind ſoever, and having illuſtrated this definition in three different kinds of quantities by ſhewing how either areas, angles, or lines may be made uſe of for this purpoſe by means of hyperbolas, logarithmic ſpirals, and logarithmic curves, and demonſtrated ſuch of the properties of theſe figures as have any relation to the preſent ſubject; it will, I doubt not, be eaſy for every one fully to comprehend the meaning of the foreſaid definition of logarithms, or to form a clear and juſt conception of their general nature and fundamental property. We may now therefore without much difficulty give ſome account of the quantity ſo often mentioned by Mr. Cotes in his *harmonia menſurarum*, and by other authors ſince his time, and which is called by him the *modulus* of the ſyſtem of logarithms to which it belongs; and may likewiſe explain the method which Mr. Cotes has made uſe of to determine the logarithm of a given *ratio*. It will alſo be eaſy to explain the etymology of the word logarithms, and to obſerve how far the original ſignification of that word is connected with the idea of it given in the above definition; and to explain likewiſe the uſes of logarithms in ſhort-

ening arithmetical calculations; the particular ſyſtems of logarithms that have been invented for that purpoſe; and the conceptions given of them by the moſt celebrated authors that have wrote concerning them. All theſe particulars we now proceed to conſider in the following manner.

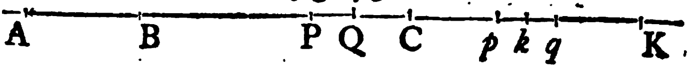
Of the quantity called by Mr. Cotes the modulus *of a ſyſtem of logarithms.*

The definition of the modulus.

219. It has been ſhewn in the foregoing articles that the parallelogram of an hyperbola is a fourth proportional to the difference of any two portions of the aſymptote whoſe *ratio* is infinitely near to a *ratio* of equality, either of the portions themſelves, and the little aſymptotic area that is the logarithm of their *ratio*; but, by the property of the aſymptotes, the parallelogram of an hyperbola is every where of the ſame magnitude; therefore this fourth proportional is every where of the ſame magnitude. In like manner it has been ſhewn that the ſubtangent of a logarithmic curve is a fourth proportional to the difference of two contiguous ordinates whoſe *ratio* is infinitely near to a *ratio* of equality, either of the ordinates themſelves, and the little abſciſs of the axis that is the logarithm of their *ratio*; and we have proved in *Art.* 202, that the ſubtangent of a logarithmic curve is every where of the ſame magnitude; therefore the fore-mentioned fourth proportional is every where of the ſame magnitude;

tude. From the analogy therefore which muſt neceſſarily ſubſiſt between all meaſures of the ſame quantities we may conclude, that in every ſyſtem of logarithms whatſoever, (whether they be areas, or lines, as in the hyperbola and logarithmic curve, or angles, or any other kind of quantity whatſoever) the fourth proportional to the difference of the terms of an infinitely ſmall *ratio*, either of the terms themſelves, and the logarithm of their *ratio* to each other, is every where, whatever be the magnitude of the terms aſſumed, of the ſame given magnitude. Thus, if a ſet of areas, lines, angles, or any other kind of quantity, be made uſe of as meaſures of the *ratios* of the ſeveral portions of the indefinite right line AP (*fig.* 75.) reckoning from the point A, to each other, the fourth proportional to the infinitely ſmall difference PQ of the terms AP, AQ, the term AP, and the area, line, angle, or other quantity, that meaſures the *ratio* of AQ to AP, will be equal to the fourth proportional to the infinitely ſmall difference *pq* of the terms A*p*, A*q*, the term A*p*, and the area, line, angle, or other correſpondent quantity, that meaſures the *ratio* of A*q* to A*p*; or, to expreſs the ſame thing accurately, if the points P, *p* be taken at pleaſure in the indefinite line AP, and the points Q, *q*, be taken very near to P and *p*, reſpectively, and be ſuppoſed to move continually towards them, ſo that

fig. 75.



the lines PQ, and *pq* ſhall be thereby diminiſhed as far as we pleaſe, the limit of the magnitude of the fourth proportional to the ſmall, but finite, difference PQ, the line AP, and the logarithm of the *ratio* of AQ to AP, will be equal to the limit of the magnitude of the fourth proportional to the ſmall, but finite, difference *pq*, the line A*p*, and the logarithm of the *ratio* of A*q* to A*p*. This limit, or (if we conſider PQ and *pq* as infinitely ſmall) this fourth proportional to, the difference of any two terms whoſe *ratio* is infinitely ſmall, (or infinitely near to a *ratio* of equality) either of the terms themſelves, and the logarithm of their *ratio*, is what Mr. *Cotes* calls the *modulus* of the ſyſtem of logarithms to which it belongs.

Derivation of the word modulus.

220. The reaſon of calling the fore-mentioned fourth proportional the *modulus* of the ſyſtem of logarithms to which it belongs, is only becauſe it characterizes and determines that ſyſtem, and diſtinguiſhes it from all other ſyſtems; the word *modulus* (which is properly a term of architecture) ſignifying no more than a meaſure, or rule, or quantity to which others are referred, and by which they are determined. Now, that this fourth proportional, which for brevity's ſake we will here call M, determines the logarithms of the ſyſtem to which it belongs, or that when M is given the logarithm of any given *ratio* in the ſyſtem of logarithms to which M belongs is given likewiſe, may be ſhewn in the

ſollowing manner. Let the logarithm of the *ratio* of any two portions of the line AP, as AQ and AP, by denoted by log. $\frac{AQ}{AP}$ or L. $\frac{AQ}{AP}$, and we ſhall have PQ : AP :: L. $\frac{AQ}{AP}$: M. Therefore 'tis evident in the firſt place, that the quantities M and L. $\frac{AQ}{AP}$ muſt always be quantities of the ſame kind, ſo that if one of them be ſuppoſed to be a ſurface, the other muſt likewiſe be a ſurface, if a line, a line, *&c.* And *2dly*, if the terms AP, AQ, and conſequently their *ratio* to each other, and that of their difference PQ to AP, are ſuppoſed to remain the ſame while either of the quantities M or L. $\frac{AQ}{AP}$ is increaſed or diminiſhed, in any proportion, the other muſt likewiſe at the ſame time be reſpectively increaſed or diminiſhed in the ſame proportion. And, as this is true of the logarithm of the infinitely ſmall *ratio* of AQ to AP, it muſt alſo be true of the logarithm of any finite *ratio* whatſoever, becauſe every ſuch *ratio* may be conſidered as a multiple of the *ratio* of AQ to AP; therefore both the kind and magnitude of the *modulus* of a ſyſtem of logarithm are determined by the kind and magnitude of the logarithms of any given *ratios* in that ſyſtem, and *vice verſâ* the kind and magnitude of the logarithms of any given *ratios* in any ſyſtem of logarithms are determined by the kind and magnitude of the *modulus* of the ſyſtem.

This

220. This has been already ſhewn to be true in the hyperbola and logarithmic curve in *Art.* 213, 214; for in theſe figures the ſubſtance of the preſent article amounts to this only; to wit, that the magnitude of the parallelogram of an hyperbola and that of the logarithm of any given *ratio* in it mutually depend upon each other, ſo that the one cannot be increaſed or diminiſhed in any proportion but the other muſt at the ſame be reſpectively increaſed or diminiſhed in the ſame proportion; and that the magnitude of the ſubtangent of a logarithmic curve, and that of the logarithm of any given *ratio* in it do in the ſame manner mutually depend upon each other, ſo that the one cannot be increaſed or diminiſhed in any proportion but the other muſt at the ſame time be reſpectively increaſed or diminiſhed in the ſame proportion. For the parallelogram of an hyperbola is the *modulus* of the ſyſtem of logarithms exhibited by its aſymptotic areas, and the ſubtangent of a logarithmic curve is the *modulus* of the ſyſtem of logarithms exhibited by the abſciſſes of its axis.

Of the ratio modularis.

221. From the foregoing article 'tis evident, that the *ratio* whereof the *modulus* of any one ſyſtem of logarithms is the meaſure, or logarithm, in that ſyſtem, is equal to the *ratio* whereof the *modulus* of any other ſyſtem of logarithms is the meaſure, or logarithm, in that other ſyſtem. This *ratio*, which Mr. Cotes calls the *ratio modularis*, and which is the ſame with

that which is meaſured by the parallelogram of an hyperbola in a ſyſtem of logarithms conſiſting of hyperbolic areas, and by the ſubtangent of a logarithmic curve in a ſyſtem of logarithms conſiſting of the abſciſſes of the axis of ſuch a curve, is that of 2.718,281,828,459, *&c*, to 1, or of 1 to .367,879,441,171, *&c.* as was obſerved in *Art.* 203.

Conjectures concerning the reaſons that probably induced Mr. Cotes to pitch upon the fore-mentioned quantity for a modulus.

222. It having been obſerved in *Art.* 220, that Mr. Cotes's reaſon for calling the quantity M the *modulus* of the ſyſtem of logarithms to which it belongs, was becauſe the logarithms of all given *ratios* in the ſyſtem were always determined by it, and this being a property that belongs equally to any other given quantity related to a ſyſtem of logarithms in a certain known manner, as for inſtance, to the logarithm of any known or given *ratio* whatſoever (as well as the *ratio* 2.718, *&c.* to 1) or to 2 M, 3 M, 4 M, $\frac{1}{2}$ M, $\frac{1}{3}$ M, or any other multiple or part of the quantity M, (for by any one of theſe quantities the logarithms of given *ratios* may be determined), it is natural to inquire why Mr. Cotes made choice of the quantity M, rather than any other given quantity, or than the logarithm of any other given *ratio*; (particularly than that of ſome remarkable *ratio*, as ſuppoſe the *ratio* of 10 to 1) for the *modulus*, or characteriſtic quantity, of the ſyſtem. Now the reaſons which determined him to this choice are, as I imagine, theſe that follow.

223. In the firſt place 'tis evident, that when either areas or lines (which are almoſt the only quantities that are the ſubject of mathematical contemplation) are made uſe of as logarithms, or meaſures, of *ratios*, and conſequently the hyperbola and logarithmic curve come under conſideration, (for though there are an infinite number of other curves that exhibit areas and lines that are meaſures of the *ratios* of certain other quantities that bear a known relation to them; ſuch for inſtance, as are the figures of tangents and ſecants, as ſhall be ſhewn hereafter, and the arches of a circle deſcribed round the center of a logarithmic ſpiral, as was obſerved in *Art.* 195; yet the hyperbola and logarithmic curve are the fitteſt of any for this purpoſe, and exhibit by their areas and abſciſſes the ſimpleſt ſyſtems of logarithms that are poſſible, inſomuch that all other areas and lines that perform the ſame office, ought as much as poſſible to be reduced to theſe figures if we would fully and clearly underſtand them) I ſay, when hyperbolas and logarithmic curves are made uſe of as meaſures of *ratios*, the quantities that moſt naturally offer themſelves to us for determining the logarithms of given *ratios* in thoſe figures are evidently the parallelogram of the hyperbola, and the ſubtangent of the logarithmic curve: and accordingly theſe quantities had always been made uſe of for this purpoſe by the writers who treated of theſe figures, and their properties as meaſures of *ratios*, before Mr. Cotes's time. 'Tis probable therefore, that a deſire of preſerving an analogy between the different ſyſtems of logarithms, and of conforming to the practice of the

writers that went before him, might be one reaſon for his chuſing that quantity for the general characteriſtic quantity of all ſyſtems of logarithms whatſoever, which in a ſyſtem of hyperbolic areas was equal to the parallelogram of the hyperbola, and in a ſyſtem of right lines exhibited by the abſciſſes of the axis of a logarithmic curve was equal to the ſubtangent of the curve; that is, for chuſing for the general characteriſtic quantity of all ſyſtems of logarithms whatſoever, the fourth proportional to the difference of any two terms whoſe *ratio* is infinitely near a *ratio* of equality, the leſſer of the terms themſelves, and the infinitely ſmall logarithm of their *ratio.*

224. Another reaſon that induced Mr. Cotes to make this choice, I conceive to have been derived from the nature of the ſolution he has given of his firſt problem, the deſign and reaſoning of which, if I underſtand it right, may be thus explained. The deſign of Mr. Cotes in this firſt problem, which he calls "*finding the meaſure of a ratio,*" I take to be this: from the mere conſideration of the terms themſelves, their differences, and their *ratios* to each other, to find certain expreſſions that ſhall be proportional to, or meaſures of, thoſe *ratios,* without firſt arbitrarily aſſuming ſome given quantity as the meaſure of a known or given *ratio,* and then finding the correſpondent meaſures of other given *ratios.* Thus let the *ratios* of AC to AB, and of AK to AB (*fig.* 75.) be any two given *ratios* whatſoever; it is required from the mere conſideration of the lines AB, AC, AK, and their parts AP, AQ, A*p*, A*q*, and differences BC, BK, PQ,

pq, to find expreſſions that ſhall be proportional to the magnitudes of theſe *ratios*.

Two principles neceſſary to be known previouſly to Mr. Cotes's ſolution of the firſt propoſition of his harmonia menſurarum.

225. In order to the underſtanding Mr. Cotes's ſolution of this problem, we muſt premiſe the following lemma; to wit, that if there be a ſet of quantities A. B. C. D. E. &c. conſiſting of any number of terms whatſoever, that are all of the ſame kind, ſo that they may be compared together, and are to each other in any proportions whatſoever; and a ſecond ſet of quantities *a. b. c. d. e.* &c. equal in number to the former, and that are all of the ſame kind one with another, though of a different kind from the quantities A. B. C. of the former ſet, be conſtituted in ſuch a manner that the terms *a. b. c. d.* &c. ſhall be analogous to the terms of the ſeries A. B. C. D. &c. or ſhall bear the ſame proportions to each other as the correſpondent terms of the ſeries A. B. C. D. &c. The ſum of any one number of terms of the ſecond ſeries, as ſuppoſe the firſt five terms *a. b. c. d. e.* ſhall bear the ſame proportion to the ſum of any other number of terms of the ſame ſeries, as ſuppoſe the firſt three terms, *a. b. c.* as the like ſums of the correſpondent terms in the firſt ſeries A. B. C. D. &c. bear to each other, or as the ſum of the firſt five terms A. B. C. D. E. of the firſt ſeries to the ſum of the firſt three terms A. B. C. of the ſame ſeries; that is, $a+b+c+d+e$ will be to $a+b+c$ as A + B + C +

$+ D + E$ is to $A + B + C$. For ſince by the ſuppoſition $a : b :: A : B$, it follows *componendo*, that $a + b : b :: A + B : B$. Again, ſince $b : c :: B : C$, and we have ſhewn that $a + b : b :: A + B : B$, it follows *ex æquo*, that $a + b : c :: A + B : C$, and *componendo*, $a + b + c : c :: A + B + C : C$. Again, ſince $c : d :: C : D$, and $a + b + c : c :: A + B + C : C$, it follows *ex æquo*, that $a + b + c : d :: A + B + C : D$, and *componendo*, $a + b + c + d : d :: A + B + C + D : D$, and *ex æquo*, $a + b + c + d : a + b + c :: A + B + C + D : A + B + C$. Again, ſince $d : e :: D : E$, it follows *ex æquo*, that $a + b + c + d : e :: A + B + C + D : E$, and *componendo* $a + b + c + d + e : a + b + c + d :: A + B + C + D + E : A + B + C + D$; but it has been ſhewn that $a + b + c + d : a + b + c :: A + B + C + D : A + B + C$; therefore *ex æquo*, $a + b + c + d + e : a + b + c :: A + B + C + D + E : A + B + C$. QED.

This demonſtration, 'tis evident, will hold good, if, inſtead of five and three terms, we compare together any other numbers of terms in each ſeries whatſoever; and conſequently will be true likewiſe when the numbers are infinite, or when the quantities compared together in each ſeries are not properly the ſums of finite numbers of thoſe terms, but the limits of ſuch ſums; which is the caſe to which it is now to be applied.

226. Another principle, that it will be neceſſary for us to prove before we enter upon the ſolution of Mr. Cotes's problem, is the following. If there be three unequal quantities, as AD, AE, AF.

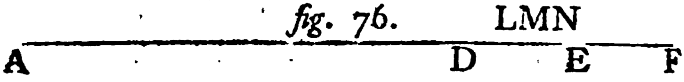


fig. 76.

(*fig.* 76.) whereof AD is the leaſt, and AF the greateſt, and their differences DE, EF, are extremely ſmall in compariſon of the quantities themſelves, the proportion of the *ratio* of AF to AD to the *ratio* of AE to AD will be very nearly equal to the proportion of the difference DF of the terms of the former *ratio* to the difference DE of the terms of the latter *ratio:* and the differences DE, DF may be taken ſo very ſmall in compariſon of AD that their proportion ſhall approach as near as we pleaſe to the proportion of the foreſaid *ratios*, or ſhall differ from it by a quantity, or *ratio*, that is leſs than any aſſigned *ratio* whatſoever. This may be proved by taking in any logarithmic curve, as *αχnr* (*fig.* 74.) the three ordinates Q*q*, R*r*, SZ reſpectively equal to the lines AD, AE, AF, and drawing the lines *rg*, and *q*G, perpendicular to SZ and meeting it in the points *g*, G. For 'tis evident the curve *q*Z will, on account of its exceſſive ſmallneſs, be very nearly a right line; conſequently the triangles ZG*q*, *rδq*, may be conſidered as rectilinear triangles ſimilar to each other, and therefore ZG will be to *rδ* very nearly in the ſame proportion as *q*G to *qδ* or as QS to QR, and conſequently as the *ratio* of SZ to Q*q* to the *ratio* of Rr to Qq: therefore the *ratio* of AF to AD will be to the *ratio* of AE to AD very nearly in the ſame proportion as the difference DF to the difference DE. And becauſe, by taking the points q, Z, continually nearer and nearer to each other, the arch qZ may be made to approach as near as we pleaſe to a right line, it follows, that the proportion of the differences ZG, $r\delta$, or DF, DE, may be made to approach as near as we pleaſe to the proportion of the logarithms qG, $q\delta$, or QS, QR, or to that of the *ratios* meaſured by thoſe logarithms, or of the *ratios* of AF to AD, and AE to AD: QED.

This might alſo be eaſily proved by the hyperbola. But if we would have an abſtract demonſtration of it, independent of both the hyperbola and logarithmic curve, and derived from the nature of *ratios*, and the definition given above of their proportion, it may be proved in the following manner. Let the *ratio* of AF to AD be divided into a very great number of leſſer *ratios* all equal to each other, by the inſertion of intermediate proportionals as AL, AM, AN, *&c.* between AD and AF. Then, ſince AM is to AL as AL to AD, it will follow *dividendo*, that LM : AL :: DL : AD, and *permutando*, LM : DL :: AL : AD; and in like manner, becauſe AN : AM :: AM : AL, we ſhall have MN : AM :: LM : AL, and *permutando*, MN : LM :: AM : AL. Therefore when AD, AL, AM, are nearly equal to each other, DL, LM, and MN will be nearly equal to each other. Therefore when the two extreme terms AF, AD, (and conſequently *a fortiori*, all the intermediate terms AL, AM, AN, *&c.* contained between

the

the points D and F) are nearly equal to each other, all the differences DL, LM, MN, &c. contained between the points D and F will be nearly equal to each other. Conſequently the difference DF will be to the difference DE very nearly in the proportion of the number of the leſſer differences DL, LM, MN, &c. contained between the points D and F to the number of thoſe differences contained, or, (if the *ratios* of AF to AD and of AE to AD are incommenſurable to each other, and conſequently no number of the differences DL, LM, &c. is exactly contained in the difference DE, to the number of thoſe differences that is) neareſt to being contained between the points D and E, or of the number of the equal *ratios* of AL to AD, AM to AL, AN to AM, &c. contained in the *ratio* of AF to AD to the number of thoſe *ratios* contained in the *ratio* of AE to AD, that is, by *Art.* 185, in the proportion of the *ratio* of AF to AD to the *ratio* of AE to AD. And as by diminiſhing continually the difference DF the proportion of AF to AD may be made to approach as near as we pleaſe to a *ratio* of equality, it follows that that difference may be taken ſo ſmall in compariſon of AD that the proportions of the differences DL, LM, MN, &c. ſhall approach as near we pleaſe to *ratios* of equality, and conſequently that the proportion of DF to DE ſhall be as nearly equal as we pleaſe to the proportion of the *ratios* of AF to AD and AE to AD. QED.

If we make uſe of the language of infiniteſimals, or infinitely ſmall quantities, we muſt ſay that when the differences DE, DF, are infinitely ſmall, their proportion to each other is accurately equal to the proportion of the *ratios* of AE to AD, and of AF to AD; or, that if there be three quantities that are infinitely near to a *ratio* of equality with each other, the exceſs of the greateſt above the leaſt will be to the exceſs of the middlemoſt above the leaſt in the ſame proportion as the *ratio* of the greateſt to the leaſt is to the *ratio* of the middlemoſt to the leaſt.

An explanation of Mr. Cotes's ſolution of the firſt problem of his harmonia menſurarum.

227. The foregoing principles being premiſed, the ſolution of the problem is as follows. Between the points B, C, take the two points P, Q, infinitely near to each other, and between the points B, K the two points p, q, infinitely near to each other. Then will the *ratios* of AQ to AP and of Aq to Ap be both of them infinitely ſmall *ratios*. Now, if theſe *ratios* are equal, that is, if the points Q and q are ſuppoſed to be taken at ſuch diſtances from P and p that Aq ſhall be to Ap as AQ to AP, it will follow, *dividendo*, that $pq : Ap :: PQ : AP$, and therefore that the expreſſion $\frac{pq}{Ap}$ is equal to the expreſſion $\frac{PQ}{AP}$, or that the $\frac{pq}{Ap}$ part of any quantity is equal to the $\frac{PQ}{AP}$ part of the ſame quantity: therefore in this caſe 'tis evident the expreſſions $\frac{pq}{Ap}$ and $\frac{PQ}{AP}$ are proportional to the *ratios* of Aq to Ap and of AQ to AP.

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The ſame will be true if the *ratios* of Aq to Ap and of AQ to AP, are unequal: for in this caſe if we take Ak to Ap as AQ to AP, 'tis evident the point k will be infinitely near the points p and q; therefore the *ratio* of Aq to Ap will be to the *ratio* of Ak to Ap, and conſequently to the *ratio* of AQ to AP, as the difference pq to the difference pk, and therefore as the expreſſion $\frac{pq}{Ap}$ to the expreſſion $\frac{pk}{Ap}$, or (becauſe Ak : Ap :: AQ AP, and conſequently pk : Ap :: PQ : AP, and $\frac{pk}{Ap} = \frac{PQ}{AP}$) $\frac{PQ}{AP}$; that is, the expreſſions $\frac{Ap}{pq}$ and $\frac{PQ}{AP}$, or the $\frac{pq}{Ap}$ and $\frac{PQ}{AP}$ parts of any given quantity of any kind and magnitude whatſoever, are proportional to, or are meaſures, or logarithms, of, the infinitely ſmall *ratios* of Aq to Ap and AQ to AP; or, if any quantity whatſoever be aſſumed at pleaſure and called M, the quantities $\frac{pq}{Ap} \times$ M, and $\frac{PQ}{AP} \times$ M, or the fourth proportionals to Ap, pq, M, and AP, PQ, M, will be meaſures, or logarithms, of the infinitely ſmall *ratios* of Aq to Ap and AQ to AP.

If therefore we conceive the points P, Q, p, q, to be infinite in number, the *ratio* of AK to AB, which is equal to the ſum of all the infinitely ſmall *ratios* of AQ to AP and Aq to Ap contained between the points B and K, will (by the foregoing lemma) be to the *ratio* of AC to AB, which is equal to the ſum of all the like infinitely ſmall *ratios* contained between the points B and C, as the ſum of all the infinitely ſmall quantities $\frac{PQ}{AP} \times M$ and $\frac{pq}{Ap} \times M$ correſponding to the *ratios* contained between the points B and K to the ſum of all the like infinitely ſmall quantities correſponding to the *ratios* contained between the points B and C; that is, the ſums of the quantities $\frac{PQ}{AP} \times M$ and $\frac{pq}{Ap} \times M$ contained between the points B and K, and B and C, are reſpectively proportional to, or are meaſures, or logarithms of, the *ratios* of AK to AB and AC to AB. The problem therefore will be ſolved, if we can compute theſe ſums; which may be performed by the well-known method of fluxions. QED.

228. The courſe of the foregoing ſolution led us to the expreſſions $\frac{pq}{Ap} \times M$ and $\frac{PQ}{AP} \times M$, or to the determining the logarithms of the infinitely ſmall *ratios* of A*q* to A*p* and AQ to AP by aſſuming a quantity at pleaſure, which we have called M, and taking ſuch portions of it as were expreſſed by fractions whoſe numerators are the differences of the terms of the *ratios*, and denominators the leſſer of the terms themſelves. 'Tis natural therefore, as this quantity will ſerve for determining and diſtinguiſhing the different ſyſtems of logarithms one from another, to make uſe of it for that purpoſe in preference to any other quantity whatſoever, and accordingly to give it the name of a *modulus*, or characteriſtic quantity: for to give this name to any other quantity that bears a known relation to it, as, for inſtance, to 2 M, 3 M, $\frac{1}{2}$ M, or $\frac{1}{3}$ M, would be introducing an unneceſſary multiplication or diviſion,

This, and the conſideration mentioned in *Art.* 223, are, as I imagine, the principal reaſons that induced Mr. Cotes to give this quantity the name of a *modulus*; which I have been the more particular in explaining, becauſe Mr. Cotes himſelf is extremely conciſe in what he ſays concerning it, and the writers ſince his time who have mentioned this word, ſeem not to have given a general definition of it.

229. From the foregoing articles 'tis evident, that, according to the language of *moduli*, which Mr. Cotes conſtantly makes uſe of in the ſeveral inſtances he has given of reducing the rectifications of curve lines, and the quadratures of curvilinear areas, and the ſolutions of other geometrical problems, to finding the meaſures of *ratios*, 'tis the ſame thing to ſay, that ſuch or ſuch an area is equal to the meaſure or logarithm, of ſuch or ſuch a *ratio* in a ſyſtem of logarithms whoſe *modulus* is ſuch or ſuch an area, as to ſay, when we are ſpeaking of an hyperbola, that ſuch or ſuch an area in equal to the aſymptotic area intercepted between any two ordinates, or portions of the aſymptote, that are in ſuch or ſuch a *ratio* to each other, in an hyperbola whoſe parallelogram is of ſuch or ſuch a magnitude; and 'tis the ſame thing to ſay, that ſuch or ſuch a line is equal to the meaſure, or logarithm of ſuch or ſuch a *ratio* is a ſyſtem of logarithms, whoſe *modulus* is ſuch or ſuch a line, as when we have recourſe to the logarithmic curve, to ſay that it is equal to the abſciſs of the axis of a logarithmic curve, whoſe ſubtangent is of ſuch or ſuch a magnitude, that is intercepted between two ordinates that are to each other in ſuch or ſuch a proportion. The

reaſon why Mr. Cotes thought fit to expreſs theſe things in a new manner rather than to keep to the old ways of expreſſing them by hyperbolas and logarithmic curves with their parallelograms and ſubtangents, which, as they were leſs abſtracted, were eaſier to underſtand, ſeems to have been chiefly becauſe he was of opinion, that the new way of expreſſing them by means of their *moduli* was better fitted than the other to reduce the ſolutions of problems to eaſy arithmetical calculations by the help of a table of logarithms; which ſeems to have been one main purpoſe of his treatiſe.

Of the modulus *of a ſyſtem of meaſures of angles,*

230 Mr. *Cotes* makes uſe of the ſame kind of expreſſions in ſpeaking of angles as when he ſpeaks of *ratios*, always conſidering circular arcs and ſectors as meaſures of angles, and diſtinguiſhing the different ſyſtems of them one from another by a quantity which he calls a *modulus.* For as the arcs of any one circle are as truly meaſures of the angles they ſubtend at its center as the ſimilar arcs of any other circle, and conſequently the magnitude of an angle is not determined by knowing only the magnitude of a circular arc that ſubtends it, unleſs we at the ſame time know the magnitude of the circle to which that arc belongs, it is neceſſary to pitch upon ſome characteriſtic quantity in every circle which varies always in the ſame proportion as the whole circumferencee, or, as the arc ſubtending any given angle at the center, varies; (ſuch as are the

whole circumference itſelf, any given portion of it, or the radius, or the ſine, tangent, ſecant, chord, or verſed ſine of any given angle whatſoever) by knowing which, and the magnitude of the arc in the ſame circle that ſubtends any propoſed angle, the magnitude of that angle will be determined. The quantity which Mr. *Cotes* makes choice of for this purpoſe, when circular arcs are made uſe of as meaſures of angles, and which he therefore calls the *modulus* of the ſyſtem of circular arcs, or meaſures of angles, to which it belongs, is the radius of the circle: in other ſyſtems of meaſures of angles, as for inſtance, in ſyſtems of circular ſectors, he defines the *modulus* of a ſyſtem to be the quantity which bears the ſame proportion to the meaſure of any given angle in that ſyſtem, as the radius of a circle bears to the arc that ſubtends the ſame angle. The reaſon of this preference given to the radius above all other quantities that would anſwer the ſame purpoſe, is to preſerve an analogy between the expreſſions uſed in ſpeaking of logarithms, or the meaſures of *ratios*, and thoſe uſed in ſpeaking of circular arcs, or the meaſures of angles: for it often happens (as was before obſerved in *Art.* 193.) that, when a problem may be conſtructed by drawing a logarithmic curve with ſuch or ſuch a line for its ſubtangent, another problem, that, in ſeveral of its conditions, reſembles the former, or that may be conſidered as only another caſe of it, may be conſtructed by drawing a circle whoſe radius is equal to the ſame line, that, in the former caſe, was made the ſubtangent of the logarithmic curve; or, according to Mr. *Cotes*'s manner of expreſſing this analogy, that, when the meaſures of *ratios* are converted into meaſures of angles, the *modulus* of the ſyſtem of the former meaſures becomes the radius of the circle that exhibits the ſyſtem of the latter meaſures. It may here be farther obſerved, that a deſire of making this analogy between the meaſures of *ratios* and the meaſures of angles appear with the more evidence, and alſo the great generality of the idea of a *modulus*, which extends to all kinds of ſyſtems of meaſures, whether conſiſting of areas or lines, (as thoſe exhibited for the menſuration of *ratios* by hyperbolas and logarithmic curves, and thoſe exhibited by circular arcs for the menſuration of angles) or any other quantities whatſoever, might be further reaſons with Mr. *Cotes*, additionally to that mentioned in *Art.* 161, for introducing the notion and name of a *modulus* into this doctrine.

Etymology of the word logarithms.

231. The proper meaning of the word logarithms, if we have regard to its etymology, is λογῶν ἀριθμοὶ, or the numbers of ſmall and equal *ratios*, or, as Dr. *Halley* calls them, *ratiunculæ*, contained in the ſeveral *ratios* of quantities one to another. Thus, if any one *ratio* taken at pleaſure, as for inſtance, the *ratio* of 10 to 1, be divided into any number of ſmall and equal parts, or leſſer *ratios*, ſuppoſe 10,000,000, by the interpoſition of intermediate proportionals between the terms of the *ratio*, the number of thoſe *ratiunculæ*, or 10,000,000th parts of the *ratio* 10 to 1, contained in any other *ratio*, as

for inſtance, in the *ratio* of 1000 to 1, to wit, 30,000,000, is, properly ſpeaking, the logarithm of that *ratio*. And becauſe, if the meaſure of the former *ratio* of 10 to 1 in any ſyſtem of meaſures whatſoever, as for inſtance, the abſciſs belonging to it in a logarithmic curve, be divided into the ſame number of equal parts as the *ratio* itſelf was divided into, to wit, 10,000,000, the meaſure, or abſciſs belonging to the latter *ratio* of 1000 to 1, will contain the ſame number of 10,000,100th parts of the meaſure, or abſciſs belonging to the *ratio* of 10 to 1, as the *ratio* of 1000 to 1 does of the like parts, or 10,000,000th parts, of the *ratio* of 10 to 1, ſo that the numbers of equal parts contained in the meaſures of theſe *ratios* will be exactly the ſame with the numbers of equal *ratiunculæ* contained in the *ratios* themſelves, theſe meaſures, or abſciſſes, have been called the logarithms of the *ratios* to which they belong. In the caſe of incommenſurable *ratios*, this ſtrict ſenſe of the word logarithms is a little extended; for, if there be two *ratios* that are incommenſurable to each other, and the former of the two be divided into any number of equal parts whatſoever, and its meaſure into the ſame number of equal parts, 'tis evident that no number whatſoever of theſe equal parts of the meaſure of the former *ratio* can be exactly equal to the meaſure of the latter *ratio*: nevertheleſs, becauſe the number of parts, into which the former *ratio*, and its meaſure, are divided, may be increaſed ſo far that the number of equal parts of its meaſure that is neareſt in magnitude to the true meaſure of the latter *ratio* ſhall approach as near as we pleaſe to that meaſure, and becauſe the celebrated inventor of logarithms, Lord *Napier* invented them merely to abridge arithmetical calculations by ſubſtituting the eaſy operations of addition and ſubtraction inſtead of the laborious ones of multiplication and diviſion, to which purpoſe it was neceſſary to repreſent the meaſures of *ratios* by numbers, or to ſuppoſe them to be divided into certain numbers of parts, accurately, when the thing is poſſible, and nearly ſo where perfect exactneſs is not to be attained, he called all the meaſures of *ratios*, whether commenſurable or incommenſurable to each other, by a name expreſſive of theſe uſes, *logarithms*. Had a more general name for them been of any importance to his deſigns, he would, no doubt, have called them by ſome other name equally extenſive with the idea ſignified by it, ſuch for inſtance as would have been *logometra*, or the meaſures of *ratios*. Thus we find that Mr. *Cotes* has entitled the firſt part of his learned treatiſe *de harmoniâ menſurarum*, in which he conſiders logarithms in their moſt extenſive ſignification, as being the meaſures of *ratios*, *logometria*, or the art of meaſuring, or finding the meaſures of, *ratios*; and throughout his whole treatiſe he has (partly, I imagine, with a view to preſerve the original ſenſe of the word logarithm, and partly to make the analogy between logarithms and circular arcs appear with the more evidence, by uſing the ſame expreſſions in ſpeaking of both, or by calling both of them *meaſures*) conſtantly called quantities that are proportional to *ratios*, the meaſures, not the logarithms, of *ratios*, as he has circular arcs, and other quantities that are proportional to angles, the meaſures of angles, and has defined logarithms in a leſs extenſive manner than is done above in *Art.* 184, to be the *numeral* meaſures of *ratios*, or to be numbers expreſſing either accurately or nearly the magnitudes of their real meaſures. But as Lord *Napier* himſelf, (who repreſents logarithms by lines which are a continued) and ſeveral other eminent writers on this ſubject, call any meaſures of *ratios*, whether commenſurable or incommenſurable to each other, (as for inſtance, the abſciſſes of the axis of logarithmic curves, and the aſymtotic areas of hyperbolas) the logarithms of the *ratios* to which they belong; and as the diſtinction between commenſurable and incommenſurable meaſures, ſeems hardly to be worth attending to, becauſe we may expreſs in numbers the magnitudes of the latter to any aſſigned degree of exactneſs; we have thought it moſt convenient to define logarithms in the moſt general manner to ſignify any meaſures of *ratios.*

Of the uſes of logarithms in abridging arithmetical calculations.

Definition of a table of logarithms.

232. The uſes of logarithms in abridging arithmetical calculations may be thus explained. Let us ſuppoſe that the values of the abſciſſes of the axis of a logarithmic curve, or the aſymptotic areas of an hyperbola, or any other meaſures, belonging to the *ratios* of all whole numbers not exceeding ſome certain number, as ſuppoſe 10,000,000, to unity, are calculated in numbers to a conſiderable degree of exactneſs,

ſuppoſe to ſeven places of figures, and ſet down in a regular manner near the numbers of whoſe *ratios* to unity they are the logarithms; ſuch a ſeries of theſe numeral expreſſions of the meaſures of *ratios* is called a table of logarithms. Now by the help of ſuch a table ready calculated to our hands we may to a ſurprizing degree abridge and facilitate the tedious operations of multiplication and diviſion, and the ſtill more laborious ones of raiſing high powers, and extracting roots.

In the firſt place therefore, becauſe the *ratio* of the product *a b* of the multiplication of any two quantities, or numbers, *a* and *b* (which let us here ſuppoſe to be whole numbers, or multiples of unity) to unity is equal to the ſum of the *ratios* of the two factors *a* and *b* to unity, it follows, that the logarithm of the *ratio* of *a b* to 1 muſt be equal to the ſum of the logarithms of the *ratios* of *a* to 1 and *b* to 1; conſequently, if we look out in the table of logarithms the logarithms of the *ratios* of *a* to 1 and *b* to 1, or (as for ſhortneſs ſake they are generally called) the logarithms of the numbers *a* and *b*, then add them together, and look out their ſum in the table, we ſhall find in the ſeries of numbers to which the logarithms correſpond, the product *a b* placed over-againſt this ſum; and thus by a ſingle eaſy addition of two logarithms together, we ſhall avoid the trouble of a multiplication.

2*dly*, In diviſion 'tis well known that the dividend is always equal to the product of the quotient and diviſor: conſequently the *ratio* of the dividend to unity is equal to the ſum of the *ratios* of the quotient and diviſor to unity, and the logarithm of the former *ratio* to the ſum of the logarithms of the two latter *ratios*, or, in other words, the logarithm of the dividend is equal to the ſum of the logarithms of the quotient and diviſor. If therefore there be two known quantities $a\,b$ and a whereof the former is to be divided by the latter to find the quotient b, we muſt look out in the table of logarithms the logarithms of ab and a the dividend and diviſor, and then ſubtract the logarithm of the diviſor a from the logarithm of the dividend $a\,b$, and the remainder will be equal to the logarithm of the quotient b, which remainder, if we look out in the table of logarithms, we ſhall find in the ſeries of numbers correſponding to the logarithms the quotient b placed over againſt it.

3*dly*, As the powers of quantities are formed by multiplying them continually into themſelves, or by taking continual proportionals to unity and the quantities themſelves, it follows that the *ratio* of any power whoſe index is m (which we will here ſuppoſe to be a whole number) of the quantity a to unity is equal to m times the *ratio* of a to unity; therefore the logarithm of the former *ratio* is equal to m times the logarithm of the latter *ratio*, or the logarithm of the quantity a^m is equal to m times the logarithm of the quantity a: If therefore the quantity a be given, and it be required to find the mth power of it, or the quantity a^m, we muſt firſt look out the logarithm of a in a table of logarithms, and then multiply it by m, and looking out the product in the table of logarithms we ſhall find in the ſeries of numbers correſponding to them the quantity a^m placed over againſt it.

4*thly*, From the laſt paragraph it follows *e converſo*, that if the quantity a^m be given, and it be required to find its *m*th root, or the quantity *a*, we muſt look out in a table of logarithms the logarithm of a^m, and divide it by the index *m*, and looking out the quotient of this diviſion in the table of logarithms, we ſhall find in the ſeries of the numbers correſponding to the logarithms the quantity *a* placed over-againſt it.

Theſe rules are from the very nature of multiplication and diviſion evidently true when *a* and *b* are whole numbers, as we have hitherto ſuppoſed them to be; but it does not appear ſo readily that they hold good likewiſe in all other caſes, that is, when either or both of the numbers *a* and *b* are mixt numbers or fractions: this therefore we ſhall now endeavour to demonſtrate, and ſhall for that purpoſe give ſome account of what is meant by negative logarithms, and of their uſe in arithmetical calculations.

Of negative logarithms.

233. The *ratios* of unequal quantities one to another, may be diſtinguiſhed into two kinds, to wit, thoſe of greater quantities to leſſer, which are called *ratios* of majority, or *majoris inæqualitatis*, ſuch for inſtance as is the *ratio* of 3 to 2, and thoſe of leſſer quantities to greater, which are called *ratios* of minority, or *minoris inæqualitatis*, as the *ratio* of 3 to 4: and in comparing theſe different kinds of *ratios* together, a *ratio* of majority is ſaid to be greater than, equal to, or leſs than, a *ratio* of minority, according as it is greater than, equal to, or leſs than, the *ratio* of majority formed by inverting the terms of the *ratio* of minority: thus, the *ratio* of 3 to 2 is ſaid to be; equal to the

ratio of 8 to 12, becauſe it is equal to the *ratio* of 12 to 8; greater than the *ratio* of 3 to 4, becauſe it is greater than that of 4 to 3; and leſs than the *ratio* of 1 to 2, becauſe it is leſs than that of 2 to 1. Now, by what Dr. *Saunderſon* has demonſtrated concerning the addition and ſubtraction of *ratios* in the 7th book of his algebra it appears,

1*ſt*, That if the *ratio* of A to B be a *ratio* of majority, and the *ratio* of C to D, be a *ratio* of minority, and leſs than the *ratio* of A to B, the ſum of the two *ratios* of A to B and C to D will be equal to the exceſs of the *ratio* of A to B above the *ratio* to D to C. For the *ratio* of C to D is added to the *ratio* of A to B, by multiplying the antecedent A into the antecedent C, and the conſequent B into the conſequent D, ſo that their ſum is equal to the *ratio* of A × C to B × D; and the *ratio* of D to C is ſubtracted from the *ratio* of A to B, by dividing the antecedent A by the antecedent D, and the conſequent B by the conſequent C, or by multiplying the antecedent A into the conſequent C, and the conſequent B into the antecedent D, ſo that the difference of theſe *ratics* is alſo equal to the *ratio* of A × C to B × D.

2*dly*, If the *ratio* of A to B be a *ratio* of majority, and the *ratio* of C to D be a *ratio* of minority as before, but greater than the *ratio* of A to B, ſo that the *ratio* of D to C cannot, properly ſpeaking, be ſubtracted from that A to B, becauſe it exceeds it, the ſum of the two *ratios* of A to B and C to D will be equal to the exceſs of the *ratio* of C to D above the *ratio* of B to A. For if the *ratio* of C to D be added to the *ratio* of A to B, the ſum will be equal to the *ratio* of A × C to B × D; and if the *ratio* of B to A be ſubtracted from that of C to D, the remaining *ratio* will be that of $\frac{C}{B}$ to $\frac{D}{A}$, or of A x C to B x D, the ſame with the foregoing ſum.

Therefore in both theſe caſes the ſum of the *ratio* of majority of A to B, and the *ratio* of minority of C to D, is equal to the exceſs of the greater of thoſe two *ratios* above a *ratio* equal and contrary to the leſſer.

Since therefore the addition of a *ratio* of minority to a *ratio* of majority has the ſame effect upon it as the ſubtraction of the contrary *ratio* of majority from it, to wit, if the *ratio* of minority be the leſſer of the two, to diminiſh it or make it a leſs *ratio* of majority, and if the *ratio* of minority be the greater of the two, to convert it into a *ratio* of minority, it follows,

3*dly*, That if we would have logarithms ſerve for diſtinguiſhing the kinds as well as the magnitudes of *ratios*, ſo that the ſame expreſſions of addition and ſubtraction that are applied to the one may alſo be applied to the other, or that the ſum of the logarithms of any two *ratios* ſhall always be equal to the logarithm of the ſum of thoſe *ratios* as well when the *ratios* are of different kinds as when they are of the ſame kind, we muſt diſtinguiſh the logarithms of the different kinds of *ratios* into different kinds, and conſider the logarithms of *ratios* of one kind as affirmative, and the logarithms of *ratios* of the other kind as negative: for if this be done, it is evident the addition of the logarithm of a *ratio* of one kind to the logarithm of a *ratio* of the oppoſite kind will produce the ſame effect as the ſubtraction of the logarithm of a *ratio* equal to the former *ratio*, but of an oppoſite kind, from the logarithm of the latter *ratio*, to wit, either to diminiſh that logarithm, or convert into a logarithm of the contrary kind; and upon theſe conſiderations mathematicians have diſtinguiſhed both, the two oppoſite kinds of *ratios*, and their logarithms, by the names of affirmative and negative, calling *ratios* of majority affirmative *ratios*, and their logarithms affirmative logarithms, (though to which kind of *ratios* and logarithms to give the name of affirmative, and which to call negative, is entirely arbitrary or matter of choice) and *ratios* of minority negative *ratios*, and their logarithms negative logarithms, and accordingly prefix the ſign + to logarithms of the former kind of *ratios*, and the ſign — to the logarithms of *ratios* of the latter kind.

4thly, According to this diſtinction of *ratios* and logarithms into affirmative and negative, 'tis evident that the logarithms of the *ratios* of all mixt numbers to unity muſt be affirmative, and the logarithms of the *ratios* of all fractions to unity muſt be negative, becauſe the former *ratios* are *ratios* of majority, and the latter *ratios* are *ratios* of minority. Therefore if we make unity the conſequent of all the *ratios* we conſider, or, which is the ſame thing, if by the logarithm of any number we always underſtand the logarithm of its *ratio* to unity, as is the uſual practice, the logarithms of all mixt numbers, or improper fractions, will be affirmative, and the logarithms of proper fractions will be negative.

234. It muſt now be ſhewn how in ſuch a table of logarithms, as is deſcribed in *Art.* 232, in which

all the whole numbers from 1 to a certain number, but no mixt numbers, or fractions, are ſet down in regular order, and their logarithms placed over-againſt them, we may find the logarithm of any given mixt number or fraction, and *vice verſa*, the mixt number or fraction, belonging to a given logarithm.

And 1*ſt*, let it be required to find the logarithm of any given proper fraction, as $\frac{m}{n}$; now here 'tis evident, that the *ratio* of 1 to $\frac{m}{n}$ is equal to the *ratio* of n to m, and therefore to the exceſs of the *ratio* of n to 1 above the *ratio* of m to 1; conſequently the logarithm of the *ratio* of 1 to $\frac{m}{n}$ is equal to the exceſs of the logarithm of the *ratio* of n to 1 above the logarithm of the *ratio* of m to 1, or to the exceſs of the logarithm of n above the logarithm of m; therefore the logarithm of $\frac{m}{n}$ is equal, but of an oppoſite kind, to the exceſs of the logarithm of n above the logarithm of m, or $\log.\frac{m}{n} =$

$-\overline{\log. n - \log. m}$, or $\log. m - \log. n$; that is, we muſt look out in the table of logarithms the logarithms of the numerator m and denominator n, and ſubtracting the former logarithm from the latter, the remainder marked with the negative ſign — will be the logarithm of the fraction $\frac{m}{n}$.

2*dly*, If $\frac{m}{n}$ be an improper fraction, or m be greater than n, its logarithm may be found as follows. The *ratio* of $\frac{m}{n}$ to 1 is equal to that of m to n, and therefore to the exceſs of the *ratio* of m to 1 above the *ratio* of n to 1; conſequently the logarithm of the *ratio* of $\frac{m}{n}$ to 1 is equal to the exceſs of the logarithm of the *ratio* of m to 1 above the logarithm of the *ratio* of n to 1, or the logarithm of the fraction $\frac{m}{n}$ is equal to the exceſs of the logarithm of the numerator m above the logarithm of the denominator n. We muſt therefore look out in the table of logarithms the logarithms of m and n, and ſubtracting the latter logarithm from the former, the remainder will be the logarithm of the fraction $\frac{n}{m}$.

'Tis evident that, if we are deſirous of uniting the ſolutions of both theſe caſes under one expreſſion, it may be done by ſaying that in both caſes, $\log. \frac{m}{n} = \log. m - \log. n$.

3*dly*, If it be required to find the proper fraction $\frac{m}{n}$ correſponding to any propoſed negative logarithm $-k$, it may be done as follows. Look along the ſeries of logarithms in the table till you find ſome logarithm (the number belonging to which call d) from which if you ſubtract k, or the logarithm of $\frac{n}{m}$ the reciprocal of $\frac{m}{n}$, the remainder will be the logarithm of a known number which call e; and we ſhall have log. $d - k =$ log. e, and $k =$ log. $d -$ log. $e =$ log. $\frac{d}{e}$, that is, log. $\frac{n}{m} =$ log. $\frac{d}{e}$; conſequently $\frac{n}{m} = \frac{d}{e}$, and $\frac{m}{n} = \frac{e}{d}$.

4thly, If it be required to find the improper fraction $\frac{m}{n}$ correſponding to any propoſed affirmative logarithm k, that lies between the logarithms of two contiguous whole numbers, it may be done as follows. Look along the ſeries of logarithms in the table till you find ſome logarithm (the number belonging to which call d) from which if you ſubtract k, the remainder will be the logarithm of a known number e, and we ſhall have log. $d - k =$ log. e, and k or log. $\frac{m}{n} =$ log. $d -$ log. $e =$ log. $\frac{d}{e}$, and conſequently $\frac{m}{n} = \frac{d}{e}$.

Note, In *Briggs*'s ſyſtem of logarithms theſe two laſt operations, which are evidently much the moſt difficult of the four operations mentioned in this article, may be performed in a much ſhorter and eaſier manner than that we have here deſcribed, and indeed with as much eaſe as the whole number correſponding to any given logarithm, that belongs to a whole number, may be found; as will be ſhewn in *Art.* 238, *Obſerv.* 5.

235. We may now apply the rules given in *Art.* 232, to the caſes where mixt numbers or fractions are concerned in arithmetical calculations.

For in the firſt place it follows from what Dr. *Saunderſon* has demonſtrated concerning the manner of adding *ratios* together, that, if we admit of the addition of *ratios* of different kinds together, or apply the word addition to the method of compounding *ratios* together, by the multiplication of antecedent into antecedent and conſequent into conſequent, as well when the *ratios* ſo compounded are of different as when they are of the ſame kind, the *ratio* of the product $a b$ of any two numbers a and b to unity, will be equal to the ſum of the *ratios* of the numbers themſelves to unity, as well when either or both the numbers a and b are mixt numbers or fractions as when they are both whole numbers. Conſequently (by *Art.* 233.) the logarithm of the *ratio* of the product $a b$ to unity will, if negative logarithms be made uſe of, be equal to the ſum of the logarithms of the *ratios* of the factors a and b to unity, or the logarithm of the product $a b$ will be equal to the ſum of the logarithms of the factors a and b. In order therefore to find the product $a b$ of any two numbers a and b of any kind whatſoever, whether whole numbers, mixt numbers, or fractions, by means of ſuch a table of logarithms as was deſcribed in *Art.* 232, we muſt proceed in the following manner: Firſt find the logarithms of a and b, (which, if they are whole numbers, may be done directly, but, if they are fractions or mixt numbers, muſt be done indirectly in the manner explained in *Art.* 234.) and, having due regard to their ſigns, add them together; then find (by the method explained in *Art.* 234.) the number correſponding to their ſum; the number ſo found will be the product $a\,b$.

2dly, Since the logarithm of $a\,b$ is $=$ log. a $+$ log. b, it follows, that log. $a\,b$ — log. a $=$ log. b; that is, if any number $a\,b$ of any kind whatſoever (whether whole number, mixt number, or fraction) is to be divided by any other number a of any kind, the quotient b may be found as follows. Firſt find the logarithms of the dividend $a\,b$ and diviſor a, and ſubtracting the latter logarithm from the former, find by *Art.* 234, the number correſponding to the remainder, the number ſo found will be the quotient b.

3dly, It follows from the nature of powers, which are only continual proportionals to unity and the firſt or ſimple power, that if a be any number whatſoever, whether whole number, mixt number, or fraction, the *ratio* of the mth power of a to unity, or of a^m to 1, will be equal to m times the *ratio* of a to 1; therefore the logarithm of the former *ratio* will be equal to m times the logarithm of a. Find therefore the logarithm of a, and multiply it by m, and the number correſponding to the product will be equal to a^m.

4thly, Since log. a^m is $= m \times$ log. a, it follows that log. a is $= \frac{\text{log. } a^m}{m}$; therefore if there be given any number a^m of any kind whatſoever, (whether whole number, mixt number, or fraction) and it be required to find its mth root a, we muſt proceed in the following manner. Firſt find the logarithm of the given number a^m; then divide this logarithm by the index m, and find the number correſponding to the quotient; the number ſo found will be equal to a.

236. Having ſhewn in the foregoing articles how the operations of arithmetic may be abridg'd by the help of any ſyſtem of logarithms whatſoever, we muſt now explain the nature of thoſe particular ſyſtems of logarithms which have been made uſe of for that purpoſe, and which are uſually known by the names of *Napier*'s logarithms and *Briggs*'s logarithms.

Definition of Napier'*s logarithms.*

If the ſubtangent of a logarithmic curve, or the parallelogram of an hyperbola, or, in general, the *modulus* of any ſet of logarithms, be called 1, or be made the ſtandard to which all the other abſciſſes of the axis of the logarithmic curve, or aſymptotic areas of the hyperbola, or, in general, all the other logarithms of the ſyſtem, are referred, and from their relation or proportion to which they are denominated, ſo that an abſciſs of the axis of the logarithmic curve that is double, triple, one half, one third, or, in general, the $\frac{m}{n}$th part, of the ſubtangent ſhall be called 2, 3, $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{m}{n}$, and, an aſymptotic area in the hyperbola that is double, triple, one half, one third, or the $\frac{m}{n}$th part, of its parallelogram, ſhall be called 2, 3, $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{m}{n}$, and in general that in any ſyſtem of logarithms, a logarithm that is double, triple, one half, one third, or the $\frac{m}{n}$th part, of the *modulus* of the ſyſtem, ſhall be called 2, 3, $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{m}{n}$; the numbers, or numerical expreſſions of the values of the logarithms, thence ariſing are called *Napier*'s logarithms.

Definition of Briggs'*s logarithms.*

If inſtead of the ſubtangent of a logarithmic curve, or the parallelogram of an hyperbola, or, in general, the *modulus* of every ſyſtem of logarithms, we chuſe the abſciſs of the axis of the logarithmic curve that is intercepted between any two ordinates that are to each other in the proportion of 10 to 1, or the aſymptotic area of the hyperbola that is intercepted between any two ordinates, or any two portions of the aſymptote, that are to each other in the proportion of 10 to 1, or, in general, in every ſyſtem of logarithms, the logarithm of the *ratio* of 10 to 1, or (as for brevity's ſake it is frequently called) the logarithm of 10, for the unit, or ſtandard to which the logarithms of all other *ratios* are referred, and from their relation or proportion to which they are denominated, ſo that a logarithm that is double, triple, one half, one third, or, in general the $\frac{m}{n}$th part of the logarithm of the *ratio* of 10 to 1 ſhall be called 2, 3, $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{m}{n}$ reſpectively; the numbers, or numerical expreſſions of the values of the logarithms, thence ariſing are called *Briggs*'s logarithms.

Theſe are the two principal ſyſtems of numerical logarithms that have hitherto been invented; there is indeed, as I have been told, a third kind of logarithms, called muſical logarithms from their being made uſe of in muſical inquiries, in which the logarithm of the *ratio* of 81 to 80, or $\frac{81}{80}$ to 1, is called 1; but theſe logarithms are entirely confined to the ſcience from which they take their name.

Some obſervations on the differences of theſe ſyſtems.

237. 'Tis evident that one and the ſame ſet of logarithms, or quantities proportional to the magnitudes of *ratios*, may, by giving the name of unity to the logarithms of different *ratios*, be made to exhibit all the different ſyſtems of numerical logarithms that can poſſibly be formed. Thus, if the parallelogram of any given hyperbola be called 1, the numerical values of the aſymptotic areas will be the logarithms of *Napier*'s ſyſtem; and, if the aſymptotic area intercepted between any two ordinates that are to each other in the proportion of 10 to 1 be called 1, the numerical values of the ſame aſymptotic areas will be the logarithms of *Briggs*'s ſyſtem; and, in the ſame manner, if the ſubtangent of a given logarithmic curve be called 1, the numerical values of the abſciſſes of the axis will be the logarithms of *Napier*'s ſyſtem; and if the

abſciſs of the axis intercepted between any two ordinates that are to each other in the proportion of 10 to 1 be called 1, the numerical values of the ſame abſciſſes of the axis will be the logarithms of *Briggs's* ſyſtem.

Hence we may obſerve in the firſt place, that when the numbers that compoſe *Napier's* and *Briggs's* logarithms are diſtinguiſhed from each other by the names of *Napier's* ſyſtem and *Briggs's* ſyſtem of logarithms, we muſt not underſtand by this expreſſion, that the quantities repreſented by *Briggs's* logarithms are neceſſarily different from thoſe repreſented by *Napier's* logarithms (it being plain from the inſtances juſt mentioned of the hyperbola and logarithmic curve, that one and the ſame ſet of quantities may be repreſented by both ſets of numbers) but we muſt take the word *ſyſtem* to relate (not to the meaſures, or logarithms themſelves, but only) to the numerical expreſſions of thoſe quantities, or logarithms, and ſo to ſignify that the ſyſtems, or ſets, of the numerical expreſſions, or names, of thoſe quantities, are different from each other, although the quantities themſelves may be the ſame.

2*dly*, Since the aſymptotic areas of an hyperbola may as well repreſent *Briggs's* logarithms as *Napier's*, it is not without a good deal of impropriety, that the latter, or *Napier's* logarithms, have by ſeveral authors been diſtinguiſhed from the former by the name of *hyperbolic* logarithms. I imagine the reaſon of giving this name to *Napier's* logarithms rather than to *Briggs's*, was becauſe *Napier's* logarithms are thoſe values of the aſymptotic areas that ariſe by calling the parallelogram of the hyperbola 1, or denominating the other areas from their relation to the parallelogram, which (though it be leſs convenient for the purpoſes of calculation than to make the area belonging to the *ratio* of 10 to 1 the unit, as ſhall be ſhewn preſently, yet) is certainly at firſt the moſt obvious and natural way of denominating them on account of the frequent and ready occurrence of this parallelogram in the contemplation of the hyperbola.

3*dly*, It is alſo an impropriety to aſſert, as ſome authors have done, that the ſubtangent of a logarithmic curve that is fitted to exhibit *Napier*'s logarithms is greater than the ſubtangent of a logarithmic curve that is fitted to exhibit *Briggs*'s logarithms; ſince any one logarithmic curve may, by giving the name of unity to different portions of the axis, be made to exhibit both thoſe ſyſtems, and all other ſyſtems whatſoever. If indeed we firſt take one logarithmic curve of any magnitude whatſoever, and call its ſubtangent 1, and then take a ſecond logarithmic curve of ſuch a magnitude, that the abſciſs of its axis correſponding to the *ratio* of 10 to 1, ſhall be equal to the ſubtangent of the former curve, and call this abſciſs 1, upon which ſuppoſitions 'tis evident, the numerical expreſſions of the abſciſſes of the axis of the former curve will be the logarithms of *Napier*'s ſyſtem, and the numerical expreſſions of the abſciſſes of the axis of the latter curve will be the logarithms of *Briggs*'s ſyſtem; I ſay, if we take two logarithmic curves related to each other in the manner here deſcribed, and give name of unity to the ſubtangent of the one, and to the abſciſs belonging to the *ratio* of 10 to 1 in the other, (which abſciſs we ſuppoſe to be equal to the ſubtangent of the former curve) the ſubtangent of the latter curve will be leſs than the ſubtangent of the former curve in the proportion of 1 (the logarithm of the *ratio* of 10 to 1 in the latter curve) to 2.302,585,092,994, 04, *&c.* (the logarithm of the ſame *ratio* in the former curve) or of .434,294,481,9 to 1: but theſe ſuppoſitions are entirely arbitrary and unneceſſary.

Of the advantage of Briggs*'s logarithms above* Napier*'s, and all other ſyſtems of logarithms whatſoever.*

238. The chief advantage of *Briggs*'s ſyſtem of logarithms above *Napier*'s, or any other ſyſtem, conſiſts in its being peculiarly adapted to the decimal notation now in uſe, as may be ſhewn in the following manner. If in performing the various operations of arithmetic none but whole numbers were ever to occur, 'twould, as I apprehend, be pretty indifferent what ſyſtem of logarithms we made uſe of: for we need only in that caſe look out in the table any given whole number, and we ſhould immediately find its logarithm, and by looking out in the table any given logarithm, we ſhould alſo directly find the whole number correſponding to it. But as fractions, either proper or improper, occur much more frequently in calculations than whole numbers; and as it is impoſſible, on account of their almoſt infinite number, to ſet down in a table the logarithms of all poſſible fractions conſiſting of five or ſeven, or any other conſiderable number of figures, that ſyſtem of logarithms will

be the moſt covenient by which we may with the greateſt eaſe find the logarithm of a given fraction, or the fraction correſponding to a given logarithm, by the help of a table of the logarithms of whole numbers only. Now 'tis evident that as all decimal fractions, whether proper or improper, have the powers of 10 for their denominators, or are form'd by dividing the whole numbers that conſtitute their numerators by ſome of the powers of 10, and conſequently their *ratios* to unity are form'd by ſubtracting the *ratios* of ſome of the powers of 10 to unity from the *ratios* of their numerators to unity, the logarithms of their *ratios* to unity will be form'd by ſubtracting the logarithms of the *ratios* of the powers of 10 to unity from the logarithms of the *ratios* of their numerators to unity, or, in other words, their logarithms will be form'd by ſubtracting the logarithms of the powers of 10 from the logarithms of their numerators. Conſequently the more ſimple the numbers are that are made uſe of as logarithms of the powers of 10, the more eaſy will it be to derive the logarithms of fractions from thoſe of whole numbers, or to find them by the help of a table that exhibits only the logarithms of whole numbers. It will therefore be *moſt eaſy* to derive the logarithms of decimal fractions from thoſe of whole numbers when the logarithms of the powers of 10 are the ſimpleſt numbers poſſible, that is, when the logarithms of 10 is call'd 1; for in this caſe the lograthim of 100, 1000, 10,000, 100,000, &c. or the logarithms of the *ratios* of 100, 1000, 10,000, 100,000, to 1, being double, triple, quadruple, quintuple, &c. of the logarithm of 10, or of the logarithm of the *ratio* of 10 to 1 (becauſe the *ratios* of 100, 1000,10,000,100,000, &c. to 1 are double, triple, quadruple, quintuple, &c. of the *ratio* of 10 to 1), will be the natural numbers 2, 3, 4, 5, &c. which are the ſimpleſt numbers and the eaſieſt to manage, that can poſſibly be found. It follows therefore that *Briggs*'s ſyſtem of logarithms, or that in which the logarithm of 10 is call'd 1, is more convenient for the purpoſes of calculation, not only than *Napiers*'s ſyſtem, but alſo, than any other ſyſtem of logarithms whatſoever. As to *Napier*'s ſyſtem, it is evidently extremely ill fitted for deriving the logarithms of decimal fractions from thoſe of whole numbers becauſe of the difficulty of managing the logarithms of 10 and its powers 100,1000, &c. ariſing from their being irrational numbers, to wit, 2.302,585, &c. (or more accurately 2.302,585,092,994,04, &c.) and 2 x 2.302,585, &c. 3 x 2.302,585,&c. or 4.605,170, &c. 6.907,755, &c. &c.

But that we may be able to explain more fully both the manner of uſing *Briggs*'s logarthims, and the convenience of that ſyſtem in calculation, we will here ſubjoin the following obſervations.

In the firſt place, in *Briggs*'s ſyſtem logarithms the integral part of every logarithm is uſually call'd its *index* or *characteriſtic*, becauſe it points out in what place of figures, reckoning from the place of tens incluſively, the correſponding number begins, or of what power of 10 the higheſt figure in it is a multiple. Thus, in the logarithm of the number 6748, to wit, 3.829,175,1, the number 3 is call'd the index, or characteriſtic, of the logarithm 3.829,1751, becauſe it ſhews that the number correſponding to this logarithm, to wit, 6748, begins in the third place of figures, reckoning from the place of tens incluſively, or that its higheſt figure to wit, 6 muſt denote a multiple of the third power of 10, or a multiple of 1000; for as the logarithm 3.829,175,1 is leſs than 4, or the logarithm of the fourth power of 10, or of 10,000, but greater than 3, or the logarithm of the third power of 10, or of 1000, the number correſponding to the logarithm 3.829,175,1, muſt be leſs than 10,000, but greater than 1000, that is, the higheſt figure in it muſt denote either 1000, or ſome multiple of 1000 leſs than the decuple, and therefore muſt be plac'd in the place of thouſands, or in the third place of figures, reckoning from the place of tens incluſively, or from the place of units excluſively. The ſame will be evident in any other inſtance.

2dly, According to this definition of an index 'tis evident the indexes of all numbers lying between 10 and 1 muſt be 0, becauſe the logarithms of all thoſe numbers being leſs than the logarithm of 10, or than 1, muſt be proper decimal fractions without any integral part at all. Thus, for inſtance, the logarithm of the number 6.748, or $\frac{6748}{1000}$, or 6 + $\frac{748}{1000}$, is .829,175,1, or 0.829,175,1. the index being 0.

3dly, 'Tis evident that in *Briggs*'s ſyſtem of logarithms the fractional parts of the logarithms of all whole, or mixt, numbers that conſiſt of the ſame ſignificant figures, are the ſame. Thus, the logarithms of the numbers, 6,748,000,674, 800, 67,480, 6748, 674.8, 67.48, and 6.748 are 6.829. 175,1, 5.829, 175,1, 4.829,175,1,3. 829,175,1,2.829,175,1,1.829,175,1,and0.829175,1, For as theſe numbers are form'd one from the other by repeated diviſions by 10, their logarithms muſt be formed one from the other by the repeated ſubtractions of an unit, (which is the logarithm of 10) which 'tis evident can only affect their indexes, or integral parts, and will leave their decimal parts as they were.

4thly, If the number 6.748, whoſe logarithm is 0.829,175,1, be divided by 10, its logarithm will be − 1 + .829,175,1, that is, the logarithm of .6748 will be — 1 + .829,175,1; if it be divided a ſecond time by 10, its logarithm muſt be again diminiſhed by 1, and will therefore be — 2 + .829,175,1, that is, the logarithm of .06748 will be — 2 + .829,175,1; if it be a third time divided by 10, its logarithm muſt be a third time diminiſhed by 1, and will therefore be — 3 + .829,175,1, that is, the logarithm of .006748 will be — 3 + .829,175,1; and ſo of all further diviſions: that is, if any propoſed decimal fraction be derived from the intermediate number between 10 and 1, that conſiſts of the ſame ſignificant figures as its numerator, by dividing that number continually by 10, and its logarithm be accordingly derived from the logarithm of that number, diminiſhing it continually by 1, and theſe diminutions, or ſubtractions, are made only from the integral parts of the logarithms, ſo as to keep the fractional part always the ſame, (that is, for example, that in ſubtracting 1 from the logarithm .829,175,1, we do not actually find the difference of thoſe two numbers, to wit, .170,824,9, and mark it with a negative ſign thus — .170,824,9, but keep them ſeparate from each other, and mark them with their proper ſigns thus, $-1 + .829{,}175{,}1$) the number of theſe ſubtractions of an unit, or (which is the ſame thing, becauſe the index of every number lying between 1 and 10 is 0) the number of units contained in the integral and negative part of the logarithm of the propoſed decimal fraction will be the ſame with the number of diviſions by 10, which the number lying between 10 and 1, and conſiſting of the ſame ſignificant figures as the propoſed decimal fraction, muſt undergo to produce that fraction, or with the number of places of figures, reckoning to the right-hand from the place of units incluſively, that muſt be filled with cyphers before the firſt ſignificant figure of the decimal fraction is to enter. Therefore when the logarithms of fractions are ſeparated in this manner into an integral negative, and a fractional affirmative part, the integral and negative parts of theſe logarithms point out to us in what places of figures the correſponding decimal fractions are to begin, or of what power of $\frac{1}{10}$ the firſt figures in their numerators are multiples, juſt in the ſame manner as the affirmative and integral parts of the logarithms of whole and mixt numbers direct us to the places of the figures of thoſe numbers. And hence it is uſual to ſeparate the logarithms of decimal fractions into two parts, an affirmative and a negative, in the manner here deſcribed, and to call the negative and integral part the index, or characteriſtic, of the logarithm, juſt as the integral part of the logarithm of a whole or mixt number is called its index or characteriſtic.

5thly, By the uſe of theſe negative indexes 'tis evident a compleat anology is preſerved between the logarithms of whole numbers, mixt numbers, and decimal fractions; ſo that the two latter kinds of numbers, and their logarithms, may with the greateſt eaſe be managed by the help of a table that exhibits only the logarithms of whole numbers: for as the fractional part of any logarithm continues always the ſame, ſo long as the ſignificant figures of the number to which it belongs continue the ſame, howſoever that number may, by dividing it continually by 10, be changed from a whole number into either a mixt number, or a proper decimal fraction, it will always ſerve to point out the ſignificant figures of the number to which it correſponds; and the index, or integral part, of the logarithm will direct us in what places thoſe figures are to be placed.

For inſtance, if in performing an arithmetical calculation by the help of a table of logarithms we ſhould meet with the logarithm 0.829,175,1, which is an affirmative logarithm, and therefore muſt belong to ſome number that is greater than unity, that is, either to ſome whole number, or ſome mixt number, we might find the number correſponding to it in the following manner. Firſt we muſt look out in the table of logarithms the logarithm whoſe fractional part is .829,175,1 (which we ſhall find either accurately, or nearly,) and over-againſt this fraction we ſhall find the figures 6748. Secondly, having found theſe figures, we muſt conſider that (ſince the logarithm 0,829,175,1 is leſs than 1 or the logarithm of 10, or) ſince the index of the logarithm 0.829,175,1 is 0, the number correſponding to it muſt be leſs than 10, but greater than 1, and therefore muſt be the mixt number 6.748.

If, inſtead of the logarithm 0.829,175,1, which belongs to a mixt number, we ſhould meet with the logarithm — 2 + .829,175,1, which is a negative logarithm, and therefore belongs to a fraction, we may find the number correſponding to it as follows. Firſt, we muſt look out the fractional part .829,175,1 of this logarithm in a table of logarithms, and over-againſt it we ſhall find the figures 6748; ſecondly, having found thoſe figures, we muſt conſider, that ſince the index of this logarithm is — 2, the firſt figure of the number correſponding to it muſt be two places below the place of units, or muſt be preceded by two cyphers including the place of units, or muſt begin in the place of hundredths; and therefore that number will be 0.06748.

And thus in *Briggs*'s ſyſtem of logarithms we may with the greateſt eaſe perform the two moſt difficult caſes of *Art*. 234, to wit, the third and fourth, in which it is required to find by means of a table of the logarithms of whole numbers the mixt number, or fraction, correſponding to a given logarithm.

As to the other two caſes of *Art*. 234, to wit, thoſe in which it is required to find the logarithm of a given mixt number, or fraction, they may be ſolved by a table of *Briggs*'s logarithms with juſt the ſame facility as the third and fourth, in the following manner.

If it be required to find the logarithm of the given mixt number 6.748, we muſt look out the number 6748, and over-againſt it we ſhall find the fractional part of its logarithm, to wit, .829,175,1; (for the indexes of logarithms, as they may be ſo eaſily ſupplied, are generally omitted in the tables of *Briggs*'s logarithms;) and putting 0 for the index of the logarithm of the number 6.748, becauſe that number lies between 1 and 10, we ſhall find this logarithm to be 0.829,175,1.

If the given mixt number had been 67.48 inſtead of 6.748, we muſt have put 1 for the index of its logarithm, which would therefore have been 1.829,175,1; if it had been 674.8, we muſt have put 2 for the index of its logarithm, which would therefore have been 2 .829,175,1.

If it be required to find the logarithm of the given decimal fraction 0.06748, we muſt look out the number 6748, and over-againſt it we ſhall find the fractional part of its logarithm, to wit, .829,175,1; and, putting — 2 for the index of the logarithm of the decimal fraction 0.06748, we ſhall find that logarithm to be — 2 + .829,175,1.

This manner of ſolving theſe two caſes, is in ſubſtance the ſame with that made uſe of in *Art.* 234, as the reader will eaſily perceive.

And here we ſhall put an end to this article, hoping that, by what has been ſaid in the courſe of it, and particularly by the laſt or 5th obſervation, and the examples ſubjoined to it, both the manner of uſing *Briggs*'s logarithms, and the great conveniency of that ſyſtem above all other ſyſtems for the purpoſes of uniting in one view the number; excepting only in one caſe, to wit, when the firſt of the four laſt figures of the logarithm is a cypher, and the three firſt figures of the fractional part of the logarithm are not ſet down over-againſt the four firſt figures of the number; and, in that caſe, the three firſt figures of the logarithm are thoſe which are ſet down in the column ſigned 0, next below the four firſt figures of the number. Thus, if I want to find the logarithm of the number 67483, I firſt look out in the ſeries of numbers the firſt four figures 6748, and then looking along the column ſigned with the figure 3, which is the laſt figure of the number 67483, I there find in the ſame line with, or over-againſt, the four figures 6748 the four laſt figures of the fractional part of the logarithm, to wit, 1944; I then look along the column ſigned 0, and next above the four figures 6748 I find the three figures 829, which are therefore the three firſt figures of the fractional part of the logarithm: therefore the fractional part of the logarithm of the number 67483 is .829,194,4, and conſequently the complete logarithm of this number is 4.829,194,4. As an example to the excepted caſe, let it be required to find the logarithm of the number 67609. Here I firſt look out in the ſeries of numbers the four firſt figures of this number, to wit, 6760, and oppoſite to them in the column ſigned 9, I find the four laſt figures of the fractional parts of its logarithm, to wit, 0045; I then look along the column ſigned 0, and I there find next below the figures 6760 the three figures 830; which are therefore the three firſt figures of the fractional part of the logarithm of the number 67609; therefore the fractional part of the logarithm of this number is .830,004,5, and conſequently its compleat logarithm is 4.830,004,5.

4thly, 'Tis evident, that the logarithms of all numbers from 1000 to 10,000, or of all numbers conſiſting of four figures, may be derived from the logarithms of the numbers of five places of figures which ariſe by multiplying them into 10, or which have the ſame ſignificant figures with the propoſed numbers of four places of figures with the addition of a cypher in the place of units: as, for inſtance, the logarithm of 6748 may be derived from that of 67480; for as they have both the ſame fractional part, we need only look out the fractional part of the latter by the directions in the laſt obſervation, which we ſhall find to be .829,175,1; and, adding 3 for the index, we ſhall have 3.829,175,1 for the logarithm of the number 6748.

5thly, As to the logarithms of numbers conſiſting of more than five figures, 'tis evidently impoſſible to find them in a direct manner by the help of *Sherwin*'s tables, which go no higher than 100,000. It is uſual therefore, when the number does not conſiſt of more than ſeven figures, to make uſe of an artifice to obtain them, which may be thus explained.

It was ſhewn in *Art.* 226, that if there be three unequal quantities that approach very nearly to an equality with each other, the exceſs of the greateſt quantity above the leaſt will be to the exceſs of the middlemoſt above the leaſt, very nearly in the ſame proportion as the *ratio* of the greateſt quantity to the leaſt is to the *ratio*

of

of the middlemoſt quantity to the leaſt, or as the logarithm of the *ratio* of the greateſt quantity to the leaſt to the logarithm of the *ratio* of the middlemoſt quantity to the leaſt. If therefore there be three whole numbers conſiſting of ſeven places of figures, or that are all greater than 1,000,000, and the greateſt of the three numbers exceeds the leaſt by only 100, which is leſs than the 10,000th part of either of the three numbers, 'tis evident, the exceſs of the greateſt of the three numbers above the leaſt will be to the exceſs of the middlemoſt number above the leaſt, very nearly in the ſame proportion as the logarithm of the *ratio* of the greateſt of the three numbers to the leaſt is to the logarithm of the *ratio* of the middlemoſt of the three numbers to the leaſt; or, which is the ſame thing, as the exceſs of the logarithm of the *ratio* of the greateſt of the three numbers to unity above the logarithm of the *ratio* of the leaſt of the three numbers to unity is to the exceſs of the logarithm of the *ratio* of the middlemoſt number to unity above the logarithm of the *ratio* of the leaſt number to unity; or, according to the uſual conciſe manner of expreſſion, as the exceſs of the logarithm of the greateſt number above the logarithm of the leaſt number to the exceſs of the logarithm of the middlemoſt number above the logarithm of the leaſt number, or as the difference of the logarithms of the greateſt and leaſt numbers to the difference of the logarithms of the middlemoſt and leaſt numbers. By this proportion we may find the logarithm of a number conſiſting of ſix or ſeven figures by *Sherwin*'s tables of logarithms in the following manner.

Let it firſt be required to find the logarithm of any whole number a conſiſting of ſeven figures. From this number ſubtract the number expreſſed by its two loweſt figures, and we ſhall thereby obtain a ſecond number, which call b, conſiſting likewiſe of ſeven figures, that is leſs than the given number a, but is exceeded by it by a number leſs than 100; becauſe the number expreſſed by the two loweſt figures of the number a muſt always be leſs than 100. Therefore, ſecondly, to the number b add 100, and we ſhall thereby obtain a third nnmber $b +$ 100 that is greater than the given number a. Therefore b, a, and $b + 100$ are three ſuch numbers as were juſt now deſcribed, that are all greater than a million, and the difference of the greateſt and leaſt equal to 100. Further, becauſe the number b was derived from the number a by taking from it the number expreſſed by its two laſt figures, 'tis evident the two laſt or loweſt places of the number b will be filled with cyphers, and conſequently the two loweſt places of the number $b + 100$ will alſo be filled with cyphers: therefore the numbers b and $b + 100$ may be produced by multiplying the numbers repreſented by their five higheſt figures by 100, and conſequently their logarithms will be equal to the logarithms of thoſe two numbers increaſed by 2, or the logarithm of 100; conſequently the logarithms of b and $b + 100$ may be found by *Obſ.* 3. Find therefore the logarithms of b and $b + 100$, and ſay as $b + 100 - b$, or 100, $: a - b ::$ log. $\overline{b + 100}$ — log. b to a fourth quantity; and that fourth quantity will be equal to log. a — log. b, and therefore being added to the log. b will give the log. a, which was to be found.

Thus,

Thus, for example, if the number whoſe logarithm is to be found, is = 5,423,758, I firſt ſubtract from it the number 58, and ſo obtain the number 5,423,700, which conſiſts likewiſe of ſeven figures, and falls ſhort of the given number 5,423,758 by leſs than 100. To this number 5,423,700 I add 100, and ſo obtain a third number 5,423,800, which is greater than the given number 5,423,758. I then conſider that the numbers 5,423,700 and 5,423,800 are reſpectively equal to 100 × 54,237 and 100 × 54,238, and therefore their logarithms are reſpectively equal to 2 + log. 54,237 and 2 + log. 54,238, that is, (as will appear by looking out the logarithms of 54,237 and 54,238 in the tables in the manner deſcribed in *Obſ.* 3.) to 2 + 4.734,295,7 and 2 + 4.734,303,7, or 6.734,295,7 and 6.734,303,7; and having found theſe logarithms, I apply the foregoing proportion, and ſay as 5,423,800 — 5,423,700 : 5,423,758 — 5,423,700 :: (log. 5,423,800 — log. 5,423,700 : log. 5,423,758 — log. 5,423,700, or) 6.734,303,7 — 6.734,295,7 : log. 5,423,758 — 6.734,295,7, that is, as 100 : 58 :: .000,008,0 : log. 5,423,758 — 6.734,295,7; therefore log. 5,423,758 — 6.734,295,7 is =

$$\frac{58 \times .000,008,0}{100} = \frac{.0004640}{100} = .000,004,640$$

or (neglecting the figures that go beyond the ſeventh place) .000,004,6, and conſequently log. 5,423,758 is = 6.734,295,7 + .000,004,6 = 6.734,300,3. QED.

Secondly, If the number, whoſe logarithm is to be found, conſiſts of ſix figures, we need only add a cypher to it, or multiply it by 10, and find the logarithm of the number of ſeven figures thence ariſing. For this logarithm diminiſhed by (the logarithm of 10, or by) an unit will be the logarithm ſought.

Thus, if the number, whoſe logarithm is to be found, is 542,375, I multiply it by 10, and ſo get the number 5,423,750 conſiſting of ſeven figures. I then proceed, as in the foregoing inſtance, to look out the logarithms of 5.423,700 and 5,423,800, which I find to be 6.734,295,7 and 6.734,303,7, and then ſay as 5,423,800 — 5,423,700 : 5,423,750 — 5,423,700 (:: log. 5,423,800 — log. 5,423,700 : log. 5,423,750 — log. 5,423,700) :: 6.734,303,7 — 6.734,295,7 : log. 5.423,750 — 6.734,295,7, that is, as 100 : 50 :: .000,008,0 : log. 5,423,750 — 6.734,295,7; therefore log. 5,423,750 — 6.734.295,7 is $= \frac{50 \times .000,008,0}{100} = .000,004,0$, and log. 5,423,750 = 6.734,2957 + .000,004,0 = 6.734,299,7, and conſequently log. 542,375 is = 5.734,299,7. QED.

But if we are deſirous of finding the logarithms of numbers conſiſting of ſix figures in a more immediate manner, and without any reference to numbers conſiſting of ſeven figures, we may obtain them by a proceſs exactly ſimilar to the foregoing, in the following manner.

Let the given number of ſix figures, whoſe logarithm is to be found, be called *a*. From this number ſubtract the number expreſſed by its loweſt figure, and we ſhall thereby obtain a ſecond number, which call *b*, that is leſs than the given number *a*, but is exceeded by it by a number leſs than 10, becauſe the number expreſſed by the loweſt figure of the number *a* muſt always be leſs than 10. Therefore, ſecondly,

condly, to the number b add 10, and we ſhall thereby obtain a third number $b + 10$ that is greater than the given number a. We have therefore the numbers b, a, and $b + 10$, that all conſiſt of ſix figures, or are all greater than 100,000, and the difference of the greateſt and leaſt is but 10, which is leſs than the 10,000th part of either of them. We may therefore apply the foregoing proportion to them with the ſame propriety as in the former caſe, to numbers conſiſting of ſeven figures, and ſay, as the difference of $b + 10$ and b the greateſt and leaſt numbers is to the difference of a and b the middlemoſt and leaſt numbers; ſo is the difference of the logarithms of $b + 10$ and b the greateſt and leaſt numbers to the difference of the logarithms of a and b the middlemoſt and and leaſt numbers, or as $10 : a - b :: \log. \overline{b + 10} - \log. b : \log. a - \log. b$. Now, becauſe the number b was derived from the number a, by taking from it the number expreſſed by its loweſt figure, 'tis evident the loweſt place of the number b will be filled with a cypher, and conſequently the loweſt place of the number $b + 10$ will alſo be filled with a cypher; therefore the numbers b and $b + 10$ may be produced by multiplying the numbers repreſented by their five higheſt figures by 10, and conſequently their logarithms will be equal to the logarithms of thoſe numbers increaſed (by the logarithm of 10, or) by 1, and therefore may be found by *Obſ.* 3. Find therefore the logarithms of b and $b + 10$, and 'tis evident the three firſt terms of the foregoing proportion will then be known, and conſequently the fourth may be diſcovered by it; that is, the value of log. *a* — log. *b* will be found by that proportion; and conſequently, if we add to that value the logarithm of *b* we ſhall thence obtain the logarithm of *a*, which was to be found.

As an example of this method of finding the logarithms of numbers conſiſting of ſix figures, we will reſume the number 542,375, uſed in the laſt inſtance, and find its logarithm by the proceſſes here deſcribed. In the firſt place then I ſubtract from this number the number 5 expreſſed by its loweſt figure, and ſo obtain the number 542,370, which conſiſts likewiſe of ſix figures, and falls ſhort of the given number 542,375 by leſs than 10. To this ſecond number I add 10, and ſo obtain a third number 542,380, which is greater than the given number 542,375. I then conſider that the numbers 542,370 and 542,380 are reſpectively equal to 10 × 54,237 and 10 × 54,238, and therefore their logarithms are reſpectively equal to 1 + log. 54,237 and 1 + log. 54,238, that is, (as will appear by looking out the logarithms of 54,237 and 54,238 in the tables in the manner deſcribed in *Obſ.* 3.) to 1 + 4·734,295,7, and 1 + 4·734,303,7, or to 5·734,295,7 and 5·734,303,7; and, having found theſe logarithms, I apply the foregoing proporion, and ſay, as 542,380 — 542,370 : 542,375 — 542,370 (:: log. 542,380 — log. 542,370 : log. 542,375 — log. 542,370 or) :: 5·734,303,7 — 5·734,295,7 : log. 542,375 — 5·734,295,7; that is, as 10 : 5 :: .000,0080 : log. 542,375 — 5·734,295,7; therefore log. 542,375 — 5·734,295,7 is $= \frac{5 \times .000,008,0}{10} = .000,004,0$, and log. 542,375 is = 5.734,295,7 + .000,004,0 = 5.734,299,7. QED.

Note, In the foregoing obſervations, we have all along ſuppoſed the numbers whoſe logarithms are ſought to be whole numbers; becauſe, by what was ſhewn in *Art.* 238, the logarithms of all other decimal numbers, whether mixt numbers or fractions, may be derived from theſe by changing only the indexes. Thus, if we have found the logarithm of the whole number 5,423,758 to be 6.734,300,3, we may eaſily derive the logarithms of 54.237,58 and .005, 423,758 from it, by ſubtracting in the firſt caſe 5, and in the ſecond 9, from its index 6, which will give us 1.734,300,3 for the logarithm of 54.237,58, and — 3 + .734,300,3 for the logarithm of .005,423,578.

6thly, We muſt now ſhew how when any logarithm is given to ſeven places of figures beſides the index, the number correſponding to it may be found by *Sherwin*'s tables. Now this may be done in the following manner. Look out the three firſt figures of the fractional part of the logarithm in the column ſigned 0, and the four laſt figures in the ſeveral colums ſigned 0, 1, 2, 3, 4, 5, *&c.* and you will find in one of thoſe columns either thoſe four figures themſelves, or four other figures very nearly equal to them. If you find the figures themſelves ſet down in any of thoſe columns, the number anſwering to the given logarithm will conſiſt of five ſignificant figures; whereof the four higheſt are thoſe ſet down in the column ſigned *num*, directly over-againſt, or in the ſame line with, the four laſt figures of the logarithm, and the fifth, or loweſt, figure is that with which the column that contains the four laſt figures of the logarithm is ſigned. And the places of theſe figures are determined by the index of the logarithm.

Thus, if it be required to find the number correſponding to the logarithm 4.734,295,7, we muſt firſt look out the three figures 734 amongſt the three firſt figures of the logarithms that are ſet down in the column ſigned 0, and then look out the four remaining figures 2957 in the ſeveral colums ſign'd 0, 1, 2, 3, 4, *&c.* and we ſhall find them in the column ſigned 7; and even with them in the column of numbers ſigned *Num*, we ſhall find the four figures 5423. Therefore 54237 are the five ſignificant figures of which the number that correſponds to the logarithm 4.734,295,7 conſiſts; and conſequently, as the index of that logarithm is 4, the number correſponding to it will be 54,237.

If the given logarithm be 6.734,295,7 inſtead 4.734,295,7, the number correſponding to it will be two places higher than in the former caſe, or will be 5,423,700.

If, when we have found the three firſt figures of the fractional part of the given logarithm in the given column ſigned 0, we do not find the four laſt figures of it ſet down exactly in any of the columns ſigned 0, 1, 2, 3, 4, 5, *&c.* that that is, if we do not find the given logarithm exactly, we muſt look out the two logarithms that are next greater and leſs than the given logarithm, or between which the given logarithm lies, and the two numbers correſponding to thoſe logarithms. We muſt then ſubtract the leſſer of the two tabular logarithms from the greater, and alſo from the given logarithm, and the leſſer of the two correſponding tabular numbers from the greater, and then ſay, as the difference of the two tabular logarithms is to the difference of the leſſer tabular logarithm and the given logarithm, ſo is the difference of the two correſponding tabular numbers to the difference of the leſſer tabular number and the number correſponding to the given logarithm. Therefore if the fourth quantity found by this proportion is added to the leſſer tabular number, the ſum will be equal to the number correſponding to the given logarithm, or will be the number ſought.

Thus, if it be required to find the number correſponding to the logarithm 6 734,300,3, which is not to be found exactly in tables, we muſt take the logarithms 6.734,303,7 and 6.734,295,7 that are the next greater and leſs than the given logarithm ſet down in the tables, and muſt look out the numbers that belong to them, which we ſhall find to be 5,423,800 and 5,423,700. We muſt then ſubtract the logarithm 6.734,295,7 from the logarithms, 6.734,300,3 and 6.734,303,7, which will leave the remainders .000.004,6 and .000,008,0, and the number 5,423,700 from the number 5,423,800, which will leave the remainder 100, and make the following proportion, (the grounds and reaſon of making which were explained in the laſt obſervation,) to wit, as 000,008.0 is to .000,004,6, ſo is 100 to the difference of the number 5,423,700 and the number ſought; therefore that difference is $= \frac{.000,004,6 \times 100}{.000,008,0}$

$= 46$

$=\frac{46 \times 100}{80} = \frac{460}{8} = 57.5$, or nearly 58; and conſequently the number ſought is 5423700 + 58, or 5,423,758.

If the given logarithm be 4.734,300,3 inſtead of 6.734,300, the number correſponding to it will be two places lower than in the former caſe, or will be 54237.58; if the logarithm be 2.734,300,3, the number correſponding to it will be 542.375,8; if 8.734,300,3, it will be 542,375,800; if 10.734,300,3, it will be 54,237,580,000; the foregoing methods diſcovering always the ſignificant figures of the correſponding number as far as ſeven figures, and no more; and this equally, whether the loweſt of thoſe ſeven figures belongs to the place of units, or to any other place above or below it.

Theſe are the methods by which the logarithms of given numbers interpolated between the tabular logarithms, may be diſcovered, and are the foundations of the rules given for that purpoſe in the ſecond chapter of the diſcourſe prefixed to *Sherwin*'s tables; in which he ſhews how the logarithms of numbers conſiſting of ſix or ſeven figures, and the number correſponding to a given logarithm may be found by the help of the numbers ſet down in the two laſt columns of numbers to the right-hand; the former of which columns contains the differences of the tabular logarithms, and is therefore marked with the letter D, and the latter the proportional parts of the differences of the tabular logarithms, (or the parts of thoſe differences that bear the ſame proportion to them as the differences of intermediate numbers interpolated between the correſpondent tabular numbers bear to the differences of the tabular numbers) and is therefore marked with the letters P*ts.*

Some examples of arithmetical operations performed by the help of Sherwin's *tables of* Briggs's *logarithms.*

240. To illuſtrate further the foregoing articles, and to ſhew more evidently the uſe of logarithms in abridging arithmetical calculatious, we will here ſubjoin a few examples of the ſeveral operations of arithmetic performed by the help of *Briggs*'s logarithms carried to ſeven places of figures, as they are contained in *Sherwin*'s tables.

And 1*ſt*, Let it be required to find the product of the multiplication of the two numbers 7589 and 6757. Here we muſt look out in the tables the logarithms of theſe numbers, which we ſhall find to be 3.880,184,6 and 3.829,753,9; and, adding theſe two logarithms together, we muſt look out their ſum 7.709,938,5, in the tables, and the number correſponding thereto, which (by the directions in obſ. 6, of art. 239) we ſhall find to be 51,278,870, will be the product ſought, as far as it can be expreſſed by ſeven ſignificant figures. But in this particular caſe 'tis eaſy to determine the eighth or laſt figure, for we need only conſider, that if we had actually multiplied the number 7589 by 6757, the firſt ſtep would have been to multiply 9 by 7, which would have produced 63; therefore 3 will be the loweſt figure in the product of thoſe numbers, and conſequently their exact product will be 51,278,873.

2dly, Let it be required to multiply the decimal fraction .007,34 by the decimal fraction .000, 876. Here we muſt look out in the tables the logarithms of theſe fractions, which we ſhall find to be — 3 + .865,696,1 and — 4 + .942, 504,1; and, adding theſe logarithms together, we muſt look out their ſum (— 6 + 1.808,200, 2 or) — 5 + .808,200,2 in the tables, and the number correſponding thereto, which we ſhall find to be .000,006,429,84, will be the product ſought.

3dly, Let it be required to find the quotient that ariſes by dividing the number 78596 by 278. Here we muſt look out in the tables the logarithms of theſe numbers, which we ſhall find to be 4.895,400,4 and 2 444,044,8; and, ſubtracting the latter logarithm from the former, we muſt look out the remainder 2.451,355,6 in the tables, and the number correſponding thereto, which we ſhall find to be 282.7194, will be the quotient ſought.

4thly, Let it be required to divide the decimal fraction .347 by the decimal fraction .00478. Here we muſt firſt look out in the tables the logarithms of theſe fractions, which we ſhall find to be — 1 + .540,329,5 and — 3 + .679,427,9; and then ſubtracting the latter logarithm from the former, we muſt look out the remainder 1.860,901,6 in the tables; and the number correſponding thereto, which we ſhall find to be 72.594,15, will be the quotient ſought.

5thly, Let it be required to divide the decimal fraction .00478 by the decimal fraction .347, which is greater than .00478. Here, as before, we muſt look out in the tables the logarithms of theſe fractions, which we ſhall find to be $-3+.679,427,9$ and $-1+.540,329,5$; and, ſubtracting the latter of theſe logarithms from the former, we muſt look out the remainder $-2+.139,098,4$ in the tables, and the number correſponding thereto, which we ſhall find to be .01377522, will be the quotient ſought.

6thly, Let it be required to find the 600th power of 1.05. Here we muſt look out in the tables the logarithm of 1.05, which we ſhall find to be 0.021,189,3; and, multiplying this logarithm by 600, we muſt look out the product 12.7135800 in the tables, and the number correſponding thereto, which we ſhall find to be 5,171,065,000,000, will be the power ſought.

This example contains a ſolution of the following queſtion, to wit, how much a ſum of money will be increaſed in a courſe of 600 years by lending it at the intereſt of 5 *per cent. per an.* and at the end of every year adding the intereſt to the principal, and lending the whole new principal thence ariſing at the ſame intereſt. For if the original ſum be called *a*, the ſecond ſum (or that produced by the addition of the intereſt to principal at the end of the firſt year) *b*, the third ſum *c*, the fourth *d*, the fifth *e*, and ſo on, 'tis evident that, as the intereſt of the firſt ſum *a* is equal to $\frac{5}{100}\times a$, or $\frac{a}{20}$, that of the ſecond ſum *b* to $\frac{5}{100}\times b$, or $\frac{b}{20}$, that of the third ſum *c* to $\frac{5}{100}\times c$, or $\frac{c}{20}$, that of the fourth ſum *d* to

$\frac{5}{100} \times d$, or $\frac{d}{20}$, &c. the ſecond ſum will b be $= a + \frac{a}{20} = \frac{21a}{20}$, the third ſum c will be $= b + \frac{b}{20} = \frac{21b}{20}$, the fourth ſum d will be $= c + \frac{c}{20} = \frac{21c}{20}$, the fifth ſum e will be $= d + \frac{d}{20} = \frac{21d}{20}$, and ſo of all the reſt; that is, every new ſum will be to the foregoing in the ſame proportion as 21 to 20. or the ſeveral ſums a, b, c, d, e, &c. at the ends of the ſeveral years are a ſeries of continual proportionals. Therefore if the firſt ſum be called 1, and the ſecond b, the following ſums will be b^2, b^3, b^4, &c. or will be the ſeveral powers of the ſecond ſum b, and conſequently the laſt ſum, or that produced at the end of the 600 years, will be the 600th power of the ſecond term b. We may therefore conclude, that if any ſum of money be called 1, the ſum that will be produced in 600 years by laying it out at compound intereſt at 5 *per cent.* in the manner here deſcribed will be (as nearly as can be expreſſed by ſeven ſignificant figures) equal to 5,171,065,000,000 times the original ſum.

Thus, for inſtance, if the original ſum be a farthing, the ſum thence produced by compound intereſt at 5 *per cent.* in 600 years will be 5,171,065,000,000 farthings, or (if we divide this number by 960, which is the number of farthings contained in a pound ſterling) 5,386,526,042 pounds ſterling. And it is eaſy to perceive that, by the ſame kind of reaſoning as was made uſe of in the preſent inſtance, all other caſes whatſoever of compound intereſt may be reduced to the finding any propoſed power of a given quantity, and may accordingly be ſolved by a table of garithms.

7thly, Let it be required to find the 600th root of the number 5,171,065,000,000. Here we muſt look out in the tables the logarithm of 5,171,065,000,000, which we ſhall find to be 12.713,580,0; and, dividing the logarithm by 600, we muſt look out the quotient .021,189,3 in the tables, and the number correſponding thereto, which we ſhall find to be 1.05, will be the root ſought.

This example contains a ſolution of the reverſe of the foregoing queſtion in compound intereſt, namely, how much a ſum of money called 1 will be increaſed in one year, if by annually adding the intereſt to the principal it increaſes in a courſe of 600 years to 5,171,065,000,000, or 5,171,065,000,000, times it's original value: for from this example it appears that it's value at the end of the firſt year will be 1.05, or $\frac{105}{100}$ of it's original value, and (conſequently) the rate of intereſt 5 *per Cent.*

The great facility with which theſe very high powers and roots are obtained by the help of logarithms, affords a ſtriking proof of their uſefulneſs in abridging calculations, ſince without them the labour of performing theſe operations would be ſo exceeding great that they would almoſt be impracticable.

8*thly*, Let it be required to find the 6th power of the fraction .05. Here we muſt look out in the tables the logarithm of this fraction, which we ſhall find to be — 2 + .698,970,0; and, multiplying this logarithm by 6, we muſt look out the product — 8 + .193,820,0 in the tables; and the number correſponding thereto, which we ſhall find to be .000,000,015,625, will be the power ſought.

9*thly*, Let it be required to find the 8th power of the decimal fraction .05. Here we muſt look out in the tables the logarithm of .05, which we ſhall find to be — 2 + .698,970,0; and, multiplying this logarithm by 8, we muſt look out the product — 11 + .591,760,0 in the tables, and the number correſponding thereto, which we ſhall find to be .000,000,000,039, 062,49, will be the power ſought.

10*thly*, Let it be required to find the ſquare root of the number 5. Here we muſt look out in the tables the logarithm of the number 5, which we ſhall find to be 0.698,970,0; and, diving this logarithm by 2, we muſt look out the quotient .349,485,0 in the tables, and the number correſponding thereto, which we ſhall find to be 2.236,068, will be the root ſought.

11*thly*, Let it be required to extract the cube root of the fraction $\frac{1}{2}$, or .5. Here we muſt look out in the tables the logarithm of the fraction .5, which we ſhall find to be — 1 + .698, 970,0; and, dividing this logarithm by 3, we muſt look out the quotient (which will be =

$-\frac{1}{3}+\frac{.698,970,0}{3}=-.333,333,3+.232,990,0=-.100,343,3=)-1+.899,656,7,$ and the number correſponding thereto, which we ſhall find to be .793,705,4, will be the root ſought.

Recapitulation of what has been ſaid in the foregoing articles concerning numerical logarithms.

241. Having now explained the nature and moſt important properties of numerical logarithms, and particularly thoſe of *Briggs*'s ſyſtem, as far as I apprehended was neceſſary towards forming a clear and juſt conception of them, and this being done at great length that no difficulties might occur in the reading them, it may not be amiſs to recapitulate in few words the ſeveral particulars that have been demonſtrated concerning them, that they may with the greater eaſe be comprehended in one ſhort and general view.

In the 1*ſt* place therefore it has been ſhewn, that in arithmetical calculations wherein only whole numbers are concerned, the operations of multiplication, diviſion, raſing powers, and extracting roots, may be performed by the addition, ſubtraction, multiplication, and diviſion of the correſponding logarithms.

2*dly*, We have explained what is meant by negative logarithms, and the deſign of introducing them, namely to diſtinguiſh the more readily from each other *ratios* of majority and *ratios* of minority.

3dly, It has been ſhewn how, if negative logarithms be admitted, we may find in a table of logarithms (that is, in a table of the logarithms of whole numbers) the logarithms of given mixt numbers and fractions, and the mixt numbers and fractions that correſpond to given logarithms.

4thly, We have ſhewn that, if negative logarithms be admitted, the ſame rules that are given for the performance of arithmetical operations by the help of logarithms when the numbers concerned in them are whole numbers hold good when thoſe numbers are mixt numbers or fractions.

Theſe four particulars are true of all ſyſtems of numerical logarithms whatſoever.

5thly, We have defined the particular ſyſtems of numerical logarithms known by the names of *Napier*'s logarithms and *Briggs*'s logarithms, and have obſerved that they are only different *names* given to quantities that are meaſures of the *ratios* to which they belong.

6thly, We have ſhewn the great excellence of *Briggs*'s ſyſtem above *Napier*'s, and all other ſyſtems whatſoever, in that it is peculiarly adapted to the decimal notation now in uſe, and have explained the analogy that is thereby preſerved between the logarithms of whole numbers, mixt numbers, and fractions, and ſhewn how from the logarithms of whole numbers we may with the greateſt eaſe derive the logarithms of mixt numbers and fractions.

7thly, We have deſcribed *Sherwin*'s logarithmic tables, and given ſeveral inſtances of the manner of finding by them the logarithms of given numbers, and the numbers of given logarithms.

And 8*thly*, We have given examples of all the different operations of arithmetic performed by the help of *Briggs*'s logarithms, as contained in *Sherwin*'s tables.

An eaſy method of performing arithmetical operations, wherein fractions are concerned, without the help of negative logarithms, by reducing them to operations by whole numbers.

242. There would now remain nothing more, as I apprehend, to be ſaid upon this ſubject, were it not that the conſideration of negative logarithms may, notwithſtanding all the endeavours that have been made to explain it, appear ſtill to ſome obſcure and perplexing. For this reaſon we ſhall now endeavour to ſhew how that conſideration may be intirely avoided, and all arithmetical operations wherein fractions are concerned, be reduced to operations by whole numbers. Now the manner of doing this will, as I conceive, be beſt explained by examples; we ſhall therefore reſume thoſe of the foregoing examples in which fractions are concerned, and perform them without the help of negative logarithms. as follows.

Let it firſt be required to multiply the decimal fraction .007,34 by the decimal fraction .000,876. Look out in the tables the logarithms of the whole numbers 734 and 876, which you will find to be 2.865,696,1 and 2.942,504,1;

and, adding theſe logarithms together, look out their ſum 5.808,200,2 in the tables; and the number correſponding thereto, which we ſhall find to be 642,984, will be the product of the whole numbers 734 and 876. But .007,34 is $= \frac{734}{100,000}$, and .000,876 is $\frac{876}{1000,000}$; therefore .00734 × .000,876 is $= \frac{734 \times 876}{100,000,000,000}$ $= \frac{642,984}{100,000,000,000} =$ .000,006,429,84. QED.

'Tis evident the ſame method of reaſoning may be applied to all other caſes whatſoever of multiplication, where either fractions, or mixt numbers, are concerned; ſo that all thoſe caſes may, if we pleaſe, be reduced to the caſe of finding by a table of logarithms the product of two whole numbers.

2dly, Let it be required to find the quotient that ariſes by dividing the decimal fraction .347 by the decimal fraction .00478. Look out in the tables the logarithms of the whole numbers 3470 and 478, which you will find to be 3.540,329,5 and 2·679,427,9; and, ſubtracting the latter logarithm from the former, look out the remainder 0.860,901,6 in the tables; and the number correſponding thereto, which we ſhall find to be 7.259,415, will be the quotient of the diviſion of 3470 by 478, or will be $= \frac{3470}{478}$. But ·347 is $= \frac{3470}{10,000}$, and .004,78 is $= \frac{478}{100,000}$;

therefore $\frac{.347}{.00478}$ is $= \dfrac{\frac{3470}{10,000}}{\frac{478}{1000}} = \frac{3470}{10,000}$

$$\times \frac{100,000}{478} = \frac{3470}{478} \times 10 = 7.259415 \times 10$$

$= 72594,15.$ QED.

Note, In this laſt proceſs, the whole number 3470 was choſen for a dividend rather than the whole number 347, that the dividend might be greater than the diviſor, and conſequently that the quotient might be greater than unity, and ſo have an affirmative logarithm. And it is evident that it is equally eaſy to deduce the quotient of the fractions from that of the whole numbers in either caſe.

3*dly*, Let it be required to divide the decimal fraction .00478 by the decimal fraction .347 which is greater than .00478. Look out in the tables the logarithms of the whole numbers 478 and 347, which you will find to be 2.679,427,9 and 2.540,329,5; and, ſubtracting the latter logarithm from the former, look out the remainder 0.139,098,4 in the tables; and the number correſponding thereto, which we ſhall find to be 1.377,522, will be the quotient of the diviſion of 478 by 347, or will be $= \frac{478}{347}$. But .00478 is $\frac{478}{100,000}$, and .347 is $= \frac{347}{1000}$; therefore

.00478

$\frac{.00478}{.347}$ is $= \frac{\frac{478}{100,000}}{\frac{347}{1000}} = \frac{478}{100,000} \times \frac{1000}{347} =$

$\frac{478}{100 \times 347} = \frac{1.377,522}{100} = .013,775,22.$ QED.

4thly, Let it be required to find the ſixth power of the decimal fraction .05. Look out in the tables the logarithm of the whole number 5, which you will find to be 0.698,970,0; and, multiplying this logarithm by 6, look out the product 4.193,820,0 in the tables; and the number correſponding to that product, which we ſhall find to be 15625, will be the ſixth power of 5, or will be $= 5^6$. But .05 is $= \frac{5}{100}$, and conſequently $\overline{.05}^{6}$ is $= \frac{5^6}{\overline{100}^{6}} = \frac{5^6}{1,000,000,000,000}$; therefore $\overline{.05}^{6}$ is $= \frac{15625}{1,000,000,000,000} = .000,000,015,625.$ QED.

5thly, Let it be required to find the 8th power of the decimal fraction .05. Look out in the tables the logarithm of the whole number 5, which you will find to be 0.698,970,0; and, multiplying this logarithm by 8, look out the product 5.591,760,0 in the tables; and the number correſponding to that product, which we ſhall find to be 390,624.9 will be the 8th power of 5, or $= 5^8$. But .05 being $= \frac{5}{100}$, it follows that $\overline{.05}|^{8}$ will be $= \dfrac{5^{8}}{\overline{100}|^{8}} = \dfrac{5^{8}}{10,000,000,000,000,000}$; therefore $\overline{.05}|^{8}$ is $= \dfrac{390,624.9}{10,000,000,000,000,000} = .000,000,000,039,062,49$. QED.

6*thly*, Let it be required to extract the cube-root of the decimal fraction .5. Multiply this fraction into 1000, and the product will be the whole number 500. Then look out in the tables the logarithm of 500, which you will find to be 2.698,970,0; and, dividing this logarithm by 3, look out the quotient .899,656,7 in the tables; the number correſponding to that quotient, which we ſhall find to be 7.937,054, will be the cube root of the number 500, or will be $= \sqrt[3]{500}$. But becauſe .5 is $= \dfrac{500}{1000}$, it follows that $\sqrt[3]{.5}$ is $= \dfrac{\sqrt[3]{500}}{\sqrt[3]{1000}} = \dfrac{\sqrt[3]{500}}{10}$; therefore $\sqrt[3]{.5}$ is $= \dfrac{7.937,054}{10} = .793,705,4$. QED.

Note, We multiplied the given fraction .5 into 1000 rather than 100 or 10, or made choice of the whole number 500 in this proceſs rather than 50, or 5, that the cube root of the denominator of the fraction $\dfrac{500}{1000}$, which is equal to the given fraction .5, may be one of the integral powers of 10, as 10, 100, 1000, and not a ſurd quantity, as would be the cube-root of 10 or 100.

From theſe examples 'tis eaſy to perceive that any other inſtances of arithmetical operations wherein fractions are concerned, may be performed in the ſame manner by the help of a table of logarithms without introducing the conſideration of negative logarithms, to wit, by ſubſtituting inſtead of the fractions whole, or mixt, numbers that have the ſame ſignificant figures with them, and performing the operations concerning theſe whole, or mixt numbers, and then from the numbers thence ariſing deriving thoſe which the fractions would have produced by conſidering in what proportion the former numbers muſt be greater than the latter.

Of arithmetical complements.

243. There is another artifice, though, I think, a much leſs natural one than the foregoing, that has been made uſe of to avoid the conſideration of negative logarithms, to wit, the uſe of *arithmetical complements*; of which, as they are frequently mentioned in books that treat of logarithms, the reader may perhaps be glad to have ſome account. We ſhall therefore now endeavour to explain them.

Hitherto, whenever we have uſed the conciſe and imperfect phraſe of *the logarithm of a number*, we have underſtood thereby the logarithm of its *ratio* to unity, which is the ſenſe in which that phraſe is moſt commonly taken; and in this caſe it is evident the logarithm of unity itſelf, being the logarithm of a *ratio* of equality, muſt be o. But if, inſtead of unity, we chuſe ſome fraction, or quantity leſs than unity, for the quantity to which we refer all other numbers, or which we make the common conſequent of all the *ratios* whereof other numbers are the antecedents, ſo that when we ſpeak of the logarithm of any number, we mean thereby the logarithm of the *ratio* of that number (not to unity as before, but) to this aſſumed fraction, it is evident the logarithm of unity will not now be o, but will be the logarithm of the *ratio* of unity to the aſſumed fraction; and therefore, as that *ratio* is a *ratio* of majority, will be an affirmative logarithm: and, in like manner, the logarithms of all fractions greater than the aſſumed fraction, being the logarithms of *ratios* of majority, will be affirmative logarithms. If therefore we aſſume a fraction that is leſs than any of the fractions that can occur in the arithmetical operations that are to be performed, it is evident that all the logarithms that we ſhall have occaſion to conſider in the courſe of thoſe operations will be affirmative logarithms. Upon this foundation 'tis uſual to aſſume the fraction .000, 000,0001, or $\frac{1}{10,000,000,000}$, or $\frac{1}{10^{10}}$, for the common conſequent of the *ratios* of all other numbers, when the fractions that occur in the calculations are not leſs than that fraction: and upon this ſuppoſition 'tis evident the logarithm of unity, (being the logarithm of the *ratio* of an unit to $\frac{1}{10,000,000,000}$, or of the *ratio* of

10,000,

10,000,000,000 to 1, or 10^{10} to 1, and conſequently equal to ten times the logarithm of 10 to 1) will be = 10, the logarithm of 10 will be = 11, that of 100 = 12, that of 1000 = 13, that of $\frac{1}{10}$, or .1, = 9, that of $\frac{1}{100}$, or .01, = 8, that of $\frac{1}{1000}$, or .001, = 7, and ſo on, the place of each of theſe numbers above or below the place of units being determined by the exceſs, or defect, of its logarithm, above or below 10, or the logarithm of unity. If the fractions that occur in the calculations are leſs than .000,000,000,1, or $\frac{1}{10^{10}}$, we muſt aſſume the fraction $\frac{1}{10^{100}}$, or 1 in the hundredth place below the place of units (which is vaſtly leſs than any fractions that ever occur in calculations, and therefore will always anſwer the purpoſe deſigned of producing affirmative logarithms; and if it did not, we might take a ſtill ſmaller fraction); and upon this ſuppoſition 'tis evident the logarithm of unity will be 100, and thoſe of 10,100,1000, .1, .01, .001, *&c.* will be 101,102,103,99,98,97, *&c.* the places of theſe numbers above or below the place of units being determined by the exceſſes or defects of their logarithms above or below 100, or the logarithm of unity. In general, the logarithms of all other numbers whatſoever, as well as the powers of 10 and $\frac{1}{10}$, will on the former of theſe ſuppoſitions be greater than their common logarithms by 10, and on the latter ſuppoſition by 100. And hence it is that the indexes of theſe new logarithms, when the numbers to which they belong are fractions, are called arithmetical complements; for as theſe new logarithms of fractions may be derived from their common logarithms by adding to them, or to their indexes 10, or 100, and the indexes of the common logarithms of fractions are negative, it follows that if the indexes of the new logarithms of fractions be added to the indexes of their common logarithms conſidered as affirmative; their ſums will be always equal to 10, or 100, or the indexes of the new logarithms of fractions will be the arithmetical complements of the indexes of common logarithms conſidered as affirmative, to 10, or 100. Thus, for inſtance, the common logarithm of the fraction .00478 is $-3 + .679,427,9$, and its new logarithm is upon the firſt of the foregoing ſuppoſitions $= 10 - 3 + .679,427,9 = 7.679,427,9$, and upon the ſecond ſuppoſition $= 100 - 3 + .679,427,9 = 97.679,427,9$, and 7 and 97 are the arithmetical complements of 3 (the index of the common logarithm of the fraction .00478) to 10 and 100.

Now let a and b be any two fractions whatſoever, whoſe product $a\,b$, and likewiſe the mth power of a, is greater than $\frac{1}{10^{10}}$. Then, ſince in multiplication it always holds, as well when either, or both, the factors are fractions or mixt numbers, as when they are both whole numbers, that unity is to either of the factors as the other factor is to the product, we ſhall have $1 : a :: b : ab$. Therefore the logarithm of the *ratio* of 1 to a will be equal to the logarithm of the *ratio* of b to ab, or log. $1 : a$ is $=$ log. $b : ab$; that is, log. $1 : \frac{1}{10^{10}}$ — log. $a : \frac{1}{10^{10}} =$ log. $b : \frac{1}{10^{10}}$ — log. $ab : \frac{1}{10^{10}}$, or (according to the new ſenſe in which the expreſſion of the logarithm of a number is now to be taken), log. 1. — log. $a =$ log. b — ab, whence it follows that log. $ab =$ log. a + log. b — log. 1, and log. $b =$ log. 1 + log. ab — log. a; that is, if the fractions a and b are to be multiplied together, and we are required to find their product ab, we muſt add together the logarithms of the factors a and b, and from their ſum ſubtract the logarithm of unity, and the number correſponding to the remainder thence obtained, will be the product ſought; and if we are to divide the fraction ab by the fraction a; and find the quotient b, we muſt add the logarithm of the dividend ab to the logarithm of unity; and from their ſum ſubtract the logarithm of the diviſor a, and the number correſponding to the remainder thence obtained will be the quotient ſought. Again, ſince it holds in all caſes, as well when a is a mixt number, or fraction, as when it is a whole number, that the *ratio* of unity to the mth power of the number a. or to a^m, is equal to m times the *ratio* of unity to that number itſelf; it follows that the logarithm of the *ratio* of 1 to a^m is equal to m times the logarithm of the *ratio* of 1 to a, or log. 1 : a^m is $=$ $m \times$ log. 1 : a, that is, log. 1 : $\frac{1}{10^{10}}$ — log. a^m : $\frac{1}{10^{10}}$ is $= m \times$ $\overline{\text{log. } 1 : \frac{1}{10^{10}} - \text{log. } a : \frac{1}{10^{10}}} = m \times$ log. 1 : $\frac{1}{10^{10}}$ — m log. a : $\frac{1}{10^{10}}$, or (according to the new ſenſe in which the expreſſion of the logarithm of a number is now to be taken) log. 1 — log. a^m is $= m \times$ log. 1 — $m \times$ log. a. Conſequently log. a^m is $= m \times$ log. a — $\overline{m - 1} \times$ log 1, and log. $a =$ $\frac{\text{log. } a^m + \overline{m - 1} \times \text{log. } 1}{m}$; that is, if the fraction a is to be raiſed to the mth power, we muſt multiply the logarithm of unity into $m-1$, and the logarithm of a into m, and ſubtracting the former product from the latter, the remainder will be the logarithm of the power a^m that was required; and, if the mth root of the fraction a^m be required, we muſt multiply the logarithm of unity by $m-1$, and add the product to the logarithm of a^m; and, dividing the ſum thence obtained by m, the quotient will be the logarithm of a, or the root ſought. 'Tis evident theſe rules will be the ſame, if inſtead of $\frac{1}{10^{10}}$ we chuſe $\frac{1}{10^{100}}$, or 1 divided by any other higher power of 10, for the common fraction to which we refer all other numbers, provided that fraction be leſs than $a\,b$ and a^m, or than all the numbers that occur in the calculation; for upon that condition only the foregoing reaſonings depended.

244. We will now perform the foregoing examples of arithmetical operations in which fractions are concerned by the help of theſe arithmetical complements, as follows.

Examples of operations in arithmetic performed by the help of arithmetical complements.

In the 1ſt place let it be required to multiply the decimal fraction .007,34 by the decimal fraction .000,876. Look out in the tables the logarithms of the whole numbers 734 and 876, which you will find to be 2.865,696,1, and 2.942,504,1; and ſince theſe are their common logarithms, or the logarithms of their *ratios* to unity, it follows that the logarithms of their *ratios* to $\frac{1}{10^{10}}$ will be (10 + 2.865,696,1 and 10 + 2.942,504,1, or) 12.865,696,1 and 12.942,504,1; that is, if we make the logarithm of unity = 10, the new logarithms of the whole numbers 734 and 876 will be 12.865,696,1 and 12.942,504,1. But, becauſe 734 and 876 are equal to .00734 × 100,000, and .000,879 × 1000,000, or exceed the fractions .00734 and .000876 in the proportions of 100,000 to 1 and 1000,000 to 1, and conſequently the *ratios* of 734 and 876 to $\frac{1}{10^{10}}$ exceed the *ratios* of .00734 and .000,876 to $\frac{1}{10^{10}}$ by the *ratios* of 100,000 to 1 and 1000,000 to 1, it follows that the logarithms of the *ratios* of 734 and 876 to $\frac{1}{10^{10}}$ muſt exceed the logarithms of the *ratios* of .00734 and .000876 to $\frac{1}{10^{10}}$ by the logarithms of the *ratios* of 100,000 to 1, and 1000,000 to 1, that is, by 5 and 6. Therefore log. .00734 is = log. 734 — 5 = 12.865,6961 — 5 = 7.865,696,1 and log. .000876 is = log. 876 — 6 = 12.942,504,1 — 6 = 6.942,504,1, that is, the new logarithms of the fractions .00734 and .000,876 will be 7.865,696,1, and 6.942, 504,1. Having thus obtained theſe logarithms, we muſt, according to the rule juſt now given for multiplication, add them together, and from their ſum 14.808,200,2 ſubtract the logarithm of unity, to wit, 10, and the remainder, to wit, 4.808,200,2, will be the logarithm of the product ſought. Now, becauſe the number correſponding to the common logarithm 0.808,200, 2, or to the new logarithm 10.808,200,2, is 6.429,84, and the logarithm 10.808,200,2 is greater than the logarithm 4.808,200,2 by 6, it follows that the logarithm of the *ratio* of 6.429, 84 to $\frac{1}{10^{10}}$ exceeds the logarithm of the *ratio* of the product ſought to $\frac{1}{10^{10}}$ by 6, or by the logarithm of the *ratio* of 1000,000 to 1; therefore the *ratio* 6.429,84 to $\frac{1}{10^{10}}$ is greater than the *ratio* of the product ſought to $\frac{1}{10^{10}}$ by the *ratio* of 1000,000 to 1; conſequently 6.429,84 is equal to 1000,000 times the product ſought, and therefore the product ſought is $= \frac{6.429,84}{1000,000}$ $=$ .000,006,429,84. QED.

Note, We have in this example dwelt very long upon the method of finding the new logarithms of the fractions .00734 and .000,876, and the fraction correſponding to the new logarithm 4.808,200,2, that we might keep the grounds and reaſons of that method continually in view; but theſe being now, I preſume, ſufficiently explained, it will be convenient to expreſs theſe things in a more conciſe manner, and to determine the indexes of the new logarithms of fractions at once by the number of places below the place of units, at which the firſt ſignificant figures of the fractions enter, and *vice verſa* the places of the firſt ſignificant figures of the fractions by the indexes of the logarithms, agreeably to what was obſerved in *Art.*243. And in this manner we ſhall now perform all the remaining examples, and alſo the foregoing example over again.

Let it therefore be required in the firſt place to multiply the fraction .00734 by the fraction .000,876. Here I firſt find by the tables that the fractional parts of the logarithms of theſe fractions are .865,696,1 and .942,504,1; and, becauſe the firſt ſignificant figure of the fraction .00734 enters in the third place below the place of units, and the firſt ſignificant figure of the fraction .000,876 enters in the fourth place below the place of units, I conclude that the index of the logarithm of the former fraction is 10 — 3, or 7, and the index of the latter fraction is 10 — 4, or 6. Therefore the compleat logarithms of the fractions .00734 and .000,876 are 7.865,696,1 and 6.942,504,1. The ſum of theſe logarithms is 14.808,200,2, from which if we ſubtract 10, or the logarithm of unity, there will remain 4.808,200,2 for the logarithm of the product ſought. But by the tables it will appear that the ſignificant figures of the number correſponding to the logarithm 4.808,200,2 are 642984; and, becauſe the index of this logarithm is 4, or 10 — 6, it follows that the firſt ſignificant figure of the number correſponding to it will enter in the ſixth place below the place of units: conſequently that number, or the product ſought, will be .000,006,429,84.

2dly, Let it be required to find the quotient that ariſes by dividing the decimal fraction .347 by the decimal fraction .00478. By proceeding in the ſame manner as in the foregoing example, we ſhall find the logarithms of theſe fractions to be 9.540,329,5 and 7.679,427,9. The former of theſe logarithms, being the logarithm of the dividend, muſt, according to the rule given above, be added to the logarithm of unity, to wit, 10, and from the ſum thence obtained, to wit, 19.540,329,5, we muſt ſubtract the latter logarithm 7.679,427,9, or logarithm of the diviſor .00478, and the remainder 11.860,90 ,6 will be the logarithm of the quotient ſought. Therefore that quotient is = 72.594,15.

3dly, Let it be required to divide the decimal fraction .00478 by the decimal fraction .347.

To 7.679,427,9 the logarithm of the dividend 00478 add 10 the logarithm of unity, and from the ſum 17.679,427,9 ſubtract 9.540,329,5, and the remainder 8.139,098,4 will be the logarithm of the quotient ſought, which is therefore = .013,775,22.

4thly, Let it be required to find the ſixth power of the fraction .05. The logarithm of this fraction is 8.698,970,0; multiply therefore this logarithm by 6, and 10, or the logarithm of unity, by (6—1 or) 5, and from the former product, to wit, 52.193,820,0, ſubtract the latter product, to wit, 50, and the remainder 2.193,820,0 will (by the rule above demonſtrated) be the logarithm of the power ſought, which therefore is = .000,000,015,625.

5thly, Let it be required to find the eighth power of the fraction .05. Here we muſt make the logarithm of unity = 100, becauſe the eighth power of .05 is leſs than $\frac{1}{10^{10}}$. Now if the logarithm of 1 be taken = 100, the logarithm of .05 will be = 98.698,970,0, which multiplied by 8 gives 789.591,760,0, from which, if we ſubtract (8—1 or) 7 times the logarithm of unity, or 700, the remainder 89.591,760,0 will be the logarithm of the power ſought, which therefore will be = .000,000,000,039,062,49.

6thly, Let it be required to extract the cube root of the decimal fraction .5. The logarithm of this fraction is 9.698,970,0, 10 being taken for the logarithm of unity. Therefore, according to the rule given above, we muſt multiply 10 into (3—1 or) 2, and add the product 20 to the logarithm 9.698,970,0, and then divide the ſum 29.698,970,0 thence obtained by 3; the quotient 9.899,656,7 of this diviſion will be the logarithm of the root ſought, which will therefore be = .793,705,4.

From theſe examples it is eaſy to perceive, that any other inſtances of arithmetical operations wherein fractions are concerned, may be performed in the ſame manner by the help of a table of logarithms, without introducing the conſideration of negative logarithms, to wit, by the uſe of theſe arithmetical complements, or by chuſing the ſmall fraction $\frac{1}{10^{10}}$, or $\frac{1}{10^{100}}$, inſtead of unity, for the common conſequent of the *ratios* whereof all other numbers are, when we ſpeak of their logarithms, conſidered as the antecedents. But as this ſuppoſition, by changing the language commonly made uſe of in treating of this ſubject, and annexing a new ſenſe to the imperfect expreſſion of the logarithm of a number, may create ſome kind of perplexity and confuſion, it ſeems reaſonable to conclude, that the former method of avoiding the conſideration of negative logarithms explained in *Art.* 242, by conſidering the fractions as whole numbers during the whole proceſs, and then finally diminiſhing the reſults of the operations in a due proportion, is preferable to this latter method of avoiding them by the help of arithmetical complements.

Of

Of the different manners in which logarithms have been explained by different writers.

245. We ſhall now endeavour to give ſome account of the different manners in which logarithms have been explained by ſome of the moſt eminent writers on them, and particularly of the conception given of them by Lord *Napier* their celebrated inventor.

Mr. Maclaurin's manner of explaining logarithms.

Take any line AB, (*fig.* 77.) at pleaſure; and let the line A M be ſuppoſed from being at firſt equal to the given line A B to increaſe *proportionally ad infinitum* by the motion of a point M moving from B towards D, that is, with a motion ſo accelerated, that the portions of the line B M generated by the point M in equal times, ſhall bear always the ſame proportion to the quantities to which they are added, ſo that, if B C and C M are any two portions of the line A D generated in equal times, C M ſhall be to A C as BC to AB: and let OP be a line that begins to be generated from nothing by the motion of the point P moving from O towards T, at the ſame time that the line A M begins to increaſe from being equal to the given line AB; and let the velocity of the point P be any uniform velocity. Then 'twill be evident

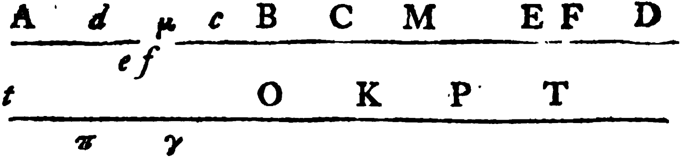
In the firſt place, that AB, AC, AM are continual proportionals, or that the *ratio* of AM to AC is equal to that of AC to AB. For ſince

CM is to AC as BC to AB, it follows, *componendo*, that AM is to AC as AC to AB.

2dly, Since BC and CM are ſuppoſed to be generated in equal times, it follows that if OK and KP are generated in the ſame times as BC and CM, they will alſo be generated in equal times, and conſequently, becauſe the velocity of the point P is uniform, will be equal to each other, and therefore will be proportional to, or meaſures, or logarithms, of, the *ratios* of AC to AB and AM to AC.

3dly, If in the line AD we take any two unequal *ratios* at pleaſure, as thoſe of AM to AC and AD to AM, and the lines KP, and PT are generated in the ſame times as the lines CM and MD, they will be proportional to, or meaſures, or logarithms, of, thoſe unequal *ratios*; that is KP will be to PT as the *ratio* of AM to AC, is to the *ratio* of AD to AM. This follows immediately from the laſt obſervation: for ſince the portions of the line OT correſponding to any two equal *ratios* are equal, it follows that the portion correſponding to a double *ratio* will be double, that to a triple *ratio* will be triple, that to a quadruple *ratio* quadruple, and that to any other multiple of the ſimple *ratio* will be the ſame multiple of the portion of OT correſponding to the ſimple *ratio*; and conſequently that the portions of the line OT correſponding to the quadruple and quintuple, or in general to the *m* and *n* multiples, of the ſimple *ratio*, will be quadruple and quintuple, or *m* fold and *n* fold of the portion correſponding to the ſimple *ratio*; that is, the portions of the line OT correſpondreſponding to any two commenſurable *ratios* whatſoever, and conſequently to any two incommenſurable *ratios* whatſoever, (for after proving this in the caſe of commenſurable *ratios* it may eaſily be ſhewn to be true likewiſe in the caſe of in commenſurable *ratios* by an *exabſurdo* demonſtration), that is, in general, the portions of the like OT correſponding to any two *ratios* whatſoever taken on the line AD, as for inſtance, the *ratios* of AM to AC and AD to AM, are reſpectively proportional to, or logarithms of, the *ratios* to which they correſpond. QED.

Fig. 77.



Again, let us now ſuppoſe that the line AM, after having been increaſed *ad infinitum*, in the manner juſt deſcribed, is made to decreaſe *proportionally ad infinitum* by the motion of the point M in a direction contrary to its former direction, ſetting out from B with the ſame velocity it had in that point when it began to move in the direction AD, that is, let it be ſuppoſed to decreaſe at ſuch a rate that the portions loſt in equal times ſhall always bear the ſame proportion to the correſpondent remainders of the line AB, ſo that, if *Bc* and *cμ* are deſcribed in equal times, B*c* ſhall be to *c*A as *cμ* to *μ*A: and let the point P move with the ſame uniform velocity from O towards *t* that it before moved with from O towards T. Then 'tis evident

In the *firſt* place that A c is to A B as A μ to A c, or that the *ratio* of Aμ to Ac is equal to the *ratio* of A c to AB. For ſince Aμ is to c μ as A c to B c, it follows, *componendo*, that A μ : A c : : A c : A B.

2dly, Since B c and c μ are ſuppoſed to be generated in equal times, it follows that if O γ and and γ π are generated in the ſame times as B c and c μ, they will alſo be generated in equal times, and conſequently, becauſe the velocity of the point P is uniform, will be equal to each other, and therefore will be proportional to, or meaſures, or logarithms, of, the *ratios* of Ac to AB and A μ to A c.

3dly, Since the portions of the line O t correſponding to equal *ratios* are equal, it follows (by the ſame kind of reaſoning as was uſed juſt now in the third obſervation concerning the portions of the line O T) that the portions of that line correſponding to any two unequal *ratios* taken on the line Ad, as for inſtance, the *ratios* of A μ to Ac and Ad to Aμ, will be reſpectively proportional to, or meaſures, or logarithms, of, thoſe *ratios*.

Laſtly, Since the velocity with which the point P deſcribes the line O t is ſuppoſed to be the ſame with that wherewith it deſcribed the line O T, it follows that the portions of the line O t generated in any times whatſoever are equal to the portions of the line O T generated in the ſame times, or that the portions of the line O t correſponding to any *ratios* whatſoever taken on the line A d are equal to the portions of the line O T correſponding to the ſame *ratios* taken on the line A D.

246. In this manner of generating logarithms 'tis evident in the firſt place, that according as the uniform velocity aſſigned to the point P is increaſed or diminiſhed in any proportion, while the velocity of the point M at the inſtant it ſets out from B towards either D or *d*, continues the ſame, the logarithms of given *ratios*, or the portions of the lines O T, O *t*, correſponding to given *ratios*, will be increaſed or diminiſhed in the ſame proportion. And hence it appears, that by varying the velocity of the point P we may form as many different ſyſtems of logarithms as we pleaſe.

2dly, 'Tis evident that, as the line A B increaſes *ad infinitum*, the velocity of the point M increaſes *ad infinitum*; and, as the line A B decreaſes *ad infinitum*, the velocity of this point will decreaſe *ad infinitum*: therefore whatever is the velocity aſſigned to the point P, or whatever is the proportion of that velocity to the velocity of the point M at the inſtant of its ſetting out from B, the velocity of the point M muſt, at ſome time or other, either in the increaſe or decreaſe of A B, be equal to the velocity of P. Now the magnitude of the line A M at the inſtant the velocity of the point M becomes equal to that of P is equal to the *modulus* of the ſyſtem of logarithms generated by the point P, or to the fourth proportional, ſo often mentioned above, to the infinitely ſmall difference of two contiguous portions of the line A M, the leſſer, or either of thoſe lines, and the infinitely ſmall difference of the correſpondent portions of the line Q P, as may be thus demonſtrated. Let

AE be the magnitude of the line AM at the inſtant the velocity of M becomes equal to that of P, and A *e* any other magnitude of A M taken at pleaſure. Take the point F a little beyond the point E, but infinitely near it, and let A *f* be to A *e* as A F to A E. From this propoſition it follows, *dividendo*, that *e f* will be to A *e* as EF to AE. But becauſe the line E F is infinitely ſmall, 'tis evident that the *ratio* of the velocities of M in the points E and F to each other will be infinitely near to a *ratio* of equality, and conſequently that the line E F is equal to the line that would have been generated in the ſame time by the velocity in E, and therefore to the line that is generated in the ſame time by the velocity of the point P, and conſequently (becauſe *e f*, and E F, being the differences of the terms of equal *ratios*, are, by the definition of the manner in which the line AM increaſes and decreaſes, deſcribed in equal times) to the line that is generated by the point P in the time of the generation of the line *e f*. Therefore as the infinitely ſmall difference *e f* of the contiguous terms A *e*, A *f*, is to the A *e* the leſſer of thoſe terms, ſo is the portion of the line OP correſponding to the difference *e f*, or the logarithm of the infinitely ſmall *ratio* of A *f* to A *e*, to the line A E. Conſequently A E is the *modulus* of the ſyſtem of logarithms deſcribed by the point P. QED.

3*dly*, It follows from the foregoing demonſtration, that the logarithms of equal *ratios* in different ſyſtems of logarithms are to each other in the ſame proportion as the *moduli* of thoſe ſyſtems, as has been ſhewn above in *Art.* 22c.

For ſince $ef : Ae :: \log. \frac{Af}{Ae} : AE$, it follows that if, while the *ratio* of Af to Ae, and conſequently that of ef to Ae continues the ſame, the third quantity of the proportion, or the logarithm of the *ratio* of Af to Ae, be increaſed or diminiſhed in any proportion, the fourth quantity A E, or the *modulus* of the ſyſtem, muſt be increaſed or diminiſhed in the ſame proportion: and therefore, as the logarithms of all equal *ratios* in different ſyſtems of logarithms bear to each other the ſame proportions, whether thoſe *ratios* be finite or infinitely ſmall, the logarithms of equal *ratios* in different ſyſtems of logarithms are to each other in the ſame proportion as the *moduli* of thoſe ſyſtems. QED.

This is the account given of logarithms by Mr. *Maclaurin* in his excellent treatiſe on fluxions, book i. chap. 6. only with this difference, that whereas we have here, for the ſake of preſerving a conſiſtency with the foregoing articles, retained our former definition of logarithms, and of the *modulus* of a ſyſtem of logarithms, to wit, that logarithms are meaſures of *ratios*, and the *modulus* of a ſyſtem of logarithms a fourth proportional to, the infinitely ſmall difference of two quantities whoſe proportion is infinitely near to a *ratio* of equality, the leſſer of thoſe quantities, and the logarithm of their *ratio* in that ſyſtem, and have then ſhewn that the portions of the lines O T, O t, are proportional to the *ratios* taken on the lines A D, A d, that correſpond to them, and therefore are the logarithms of thoſe *ratios*, and likewiſe that A E, or the magnitude of the line A M at the inſtant the velocity of M becomes equal to that of P, is equal to the fore-mentioned fourth proportional, and therefore is the *modulus* of the ſyſtem, he on the contrary defines the portions of O T and O *t* generated in any times whatever, to be the logarithms of the correſpondent portions of the lines A D, A *d*, (for he does not in his definition ſuppoſe any connection between logarithms and *ratios*,) and the line AE, or the magnitude of the line A M at the inſtant the velocity of M becomes equal to that of P, to be the *modulus* of the ſyſtem, and then proves that theſe logarithms are meaſures of the correſpondent *ratios* taken on the lines A D, A *d*, or of the *ratios* of the portions of thoſe lines whoſe differences are generated in the ſame times with theſe logarithms, and alſo that the logarithms of equal *ratios* in different ſyſtems of logarithms are proportional to the *moduli* of thoſe ſyſtems, from whence the other property of the *moduli*, to wit, their being equal to the fourth proportionals above-mentioned, may eaſily be derived.

Lord Napier's *manner of explaining logarithms.*

247. The idea of logarithms given by lord *Napier* is but a part of the foregoing; for he only ſuppoſed the line A B to decreaſe proportionally by the motion of the point M from B towards A. The reaſon of his making this ſuppoſition I take to have been this: the principal deſign of his invention being to abridge trigonometrical calculations, in which 'tis moſt uſual and convenient to determine angles by their

ſines rather than by their tangents and ſecants, he adapted it to the finding the logarithms of ſines, or of their *ratios* to the *radius*; and therefore, as all the ſines in a circle are leſs than the *radius*, he choſe to ſuppoſe a right line, from being at firſt equal to a certain given magnitude, as A B, by which he repreſented the *radius* of a circle, to decreaſe continually *ad infinitum*, and thereby become ſucceſſively equal to all the ſines of the circle, whereby the logarithms, or portions of the line O *t* generated in the ſame times with the decrements of the line A B, would exhibit the logarithms of the ſines that were equal to the remainders of that line. This, I imagine was the reaſon of his chuſing to ſuppoſe the line A M to decreaſe proportionally rather than to increaſe proportionally; and hence alſo, as in making uſe of ſines preferably to tangents and ſecants, (all which lines it is uſual to refer to, and denominate from, the *radius*) he found more frequent occaſion to conſider *ratios* of minority than *ratios* of majority, he applied the terms affirmative and negative in a different manner from later mathematicians, and called the former kind of *ratios* and their logarithms affirmative, as being the plainer and more familiar appellation, and the latter kind of *ratios* and their logarithms negative.

248. Further, as the given line A B (*fig.* 77.) was to repreſent the *radius* of a circle, which in tables of ſines and tangents is moſt generally denoted either by 1, or by ſome power of 10, or number wherein 1 is the only ſignificant figure; as 10,000, 100,000, 1000,000, lord *Napier*

pier thought proper, in conformity to this well-known cuſtom, to call it 1000,000, or to give the name of unity to the 1,000,000th part of A B; or to denominate all the portions of the lines A *d* and O *t* from their relation to the 1000,000th part of A B: and laſtly, amongſt the infinite variety of velocities that might be aſſigned to the point P, he choſe the initial velocity of the point M, or ſuppoſed the uniform velocity of P to be equal to the firſt velocity of M at the inſtant of its beginning to move from the point B. And hence aroſe his firſt ſyſtem of logarithms.

249. 'Tis evident that, as the velocity of P is here ſuppoſed to be equal to the firſt velocity of M, and conſequently the line A B, which is here called 1,000,000, is the value of A M at the inſtant the velocities of P and M are equal, and therefore is equal to the *modulus* of the ſyſtem, theſe numerical logarithms are ſuch as would ariſe by calling the *modulus* of the ſyſtem 1000,000, and therefore are not exactly the ſame with thoſe which were called by the ſame name, to wit, *Napier's* logarithms, in *Art.* 236, and which reſulted from ſuppoſing the *modulus* of the ſyſtem to be called 1; but they are equal to thoſe logarithms multiplied by 1000,000, and and therefore conſiſt of the ſame ſignificant figures, only raiſed higher towards the left-hand. But, as the ſame purpoſes in calculation are anſwered with equal convenience by both theſe ſyſtems, and by any other ſyſtems the logarithms of which conſiſt of the ſame ſignificant figures, 'tis uſual to conſider all theſe different

ſyſtems as the ſame, or to call all ſyſtems of logarithms wherein the *modulus* is denoted either by 1, or 10, or 100, or 1000, or any other power of 10, or number wherein 1 is the only ſignificant figure, by the name of *Napier*'s logarithms.

250. The reaſon why lord *Napier* choſe to denote AB by 1000,000 rather than 1, was probably that all the logarithms of his ſyſtem might be whole numbers, a circumſtance that, as whole numbers were formerly eſteemed much clearer and eaſier to conceive than fractions, was thought to be of conſiderable importance, and that induced the calculators of tables of ſines and tangents of thoſe times to denote the *radius* of a circle by 10,00c, or 100,000, or 1000,000, or ſome other high power of 10 rather than by 1, that all the ſines and other lines in the circle might be whole numbers. And the reaſon why he denoted A B by 1000,000 rather than by 10,000, or 10,000,000, or any other power of 10, was becauſe he propoſed to calculate his logarithms to ſix places of figures.

251. Theſe logarithms of *Napier*'s firſt ſyſtem, or which reſult from ſuppoſing the *modulus* of the ſyſtem to be denoted by 1, or ſome number wherein 1 is the only ſignificant figure, are ſometimes called hyperbolic logarithms, (tho' without any propriety, as has been ſhewn in *Art.* 237, becauſe any one aſymptotic area whatſoever may as well be denoted by 1, or 1000,000, or any other number wherein 1 is the only ſignificant figure, as the parallelogram of the hyperbola) and ſometimes alſo *natural* logarithms; the reaſon of which latter name I take to have been derived from the conſideration of its being

more obvious and natural to aſſign to the point P the initial velocity of the point M than any other velocity greater or leſs than that initial velocity of M.

252. In this firſt ſyſtem of numeral logarithms invented by lord *Napier*, the logarithm of the *ratio* of 10 to 1, or the portion of the line O *t* generated while A B decreaſes to a tenth part of its firſt magnitude, or (if you chuſe to ſuppoſe the line A B to increaſe rather than decreaſe) the portion of the line O T generated while A B increaſes to ten times its firſt magnitude, will be to A B as 2,302,585,092,994,0 to 1, as will appear by calculation; and therefore, if A B is called 1, will be = 2,302,585,092,994,0; if A B is called 1000,000, to 2,302,585.092,994,0; and if A B is called 10,000,000,000,000, to 23,025,850,929,940.

253. But this ſyſtem of logarithms lord *Napier* afterwards changed for another, which he found to be vaſtly more convenient for the arithmetical uſes to which he applied them, and which, as he communicated it to Mr. *Briggs* the *Savilian* profeſſor of geometry in the univerſity of Oxford, and Mr. *Briggs* joined with him in calculating a table of them, and after his death continued to proſecute the ſame deſign, are uſually known by the name of *Briggs*'s logarithms. In this ſecond ſyſtem the logarithm of the *ratio* of 10 to 1, is denoted by 1, or 1000,000, or ſome number wherein 1 is the only ſignificant figure, (for the ſyſtems ariſing from all theſe different ſuppoſitions are conſidered as the ſame ſyſtem, and are all called by the ſame name)

name) or if A B be ſtill called 1, or 1000,000, or ſome other number wherein 1 is the only ſignificant figure, the velocity of the point P is not ſuppoſed to be equal to the initial velocity of the point M, but to be leſs than that velocity in ſuch a proportion that the portion of the line O*t* generated while A B decreaſes to a tenth part of its firſt magnitude, or the portion of the line OT generated while AB increaſes to ten times its firſt magnitude, ſhall, inſtead of being greater than A B in the proportion of 2.302,585,092,994,0 to 1, be only equal to A B. And ſince it has been ſhewn above that the *moduli* of different ſyſtems of logarithms are to each other in the ſame proportion as the logarithms of equal *ratios* in thoſe ſyſtems, and the logarithm of the *ratio* of 10 to 1 is leſs in this latter than in the former ſyſtem in the proportion of 1 to 2.302,585,092,994,0, it follows that the *modulus* of this latter ſyſtem will be leſs than the *modulus* of the former ſyſtem in the proportion of 1 to 2.302,585,092,994,0, or of .434,294,481,903,2 to 1, and therefore (as A B was the *modulus* of the former ſyſtem) will be = .434,294,481,903,2 x A B. Therefore if A B is called 1, the *modulus* of *Briggs*'s ſyſtem of logarithms will be equal to .434,294,481,903,2; if A B is called 1000,000, to 434,294.481,903,2; and if A B is called 10,000,000,000,000, to 4,342,944,819,032.

254. 'Tis evident, there is by no means an abſolute neceſſity in explaining the nature of *Briggs*'s logarithms by the help of the lines A M, O P, and in ſhewing the difference between thoſe logarithms and thoſe of *Napier*'s firſt ſyſtem, to ſuppoſe the velocity of P to be different

in the one caſe from what it is in the other, or the real lengths of the logarithms, or portions of O P, correſponding to equal *ratios* in the two ſyſtems to be different from each other; but, if we pleaſe, we may in both caſes ſuppoſe the velocity of P to be equal to any given velocity whatſoever, as, for inſtance, to the initial velocity of M. For upon this ſuppoſition we need only give the name of unity, or of 1000,000, or of ſome number other wherein 1 is the only ſignificant figure, to the logarithm of the *ratio* of 10 to 1, (or to the portion of the line O P generated while A B increaſes to ten times, or decreaſes to the tenth part of its firſt magnitude) and denominate all the portions both of the line A M and the line O P by proportional numbers; and the numeral expreſſions of the portions of the line O P will be *Briggs*'s logarithms of the correſponding *ratios*. But then 'tis evident that A B will not now be called 1, or 1000,000, or any number wherein 1 is the only ſignificant figure, but being leſs than the logarithm of the *ratio* of 10 to 1, (which in this caſe is called 1, or 1000, 000, *&c.*) in the proportion of 1 to 2.302,585, 092,994,0, or of .434,294,481,903, 2 to 1, will be = .434,294,481,903,2, or 434,294.481, 903,2, *&c.* And upon this account, as the authors who have explained logarithms in this manner have generally thought it to be convenient, though not neceſſary, to give the ſame name to A B in both caſes, the other method of diſtinguiſhing theſe different ſyſtems one from another, by ſuppoſing the velocity of the point P to be diminiſhed, has been moſt commonly made uſe of.

As to the advantage of *Briggs*'s ſyſtem, or *Napier*'s ſecond ſyſtem, of logarithms above *Napier*'s firſt ſyſtem for the purpoſes of calculation, they have been already very fully explained in ſome of the former articles. We ſhall therefore ſay nothing more of them in this place.

255. We have hitherto ſuppoſed the line O P to be generated by the point P moving with an uniform velocity, and the line A M to increaſe or decreaſe proportionally: but this conception of the generation of logarithms is more confined than it need be; for if we conceive the velocity of the point P to vary in any manner whatſoever, ſo as to deſcribe equal lines in any unequal times, and at the ſame time conceive the velocity of the point M to vary in ſuch a manner that in the ſame unequal times in which the point P deſcribes equal lines the point M ſhall deſcribe proportional lines, then 'tis evident that if the points M and P ſet out from B and O with the ſame velocities as they were ſuppoſed to have in any particular ſyſtem of logarithms when the velocity of P was uniform, we ſhall have the ſame portions of the line O P belonging to, or generated in the ſame times with, any given portions of the line A M as we had in the former caſe, ſo that the portions of the former line will be, juſt as before, the meaſures or logarithms, of the *ratios* of the correſpondent portions of the latter; that is, if B C is to A B as C M to A C, and conſequently A B to A C as A C to A M, and while the point P generates the equal lines OK, KP, &c. the point M generates the lines B C, C M, *&c.* the line O T correſponding to the line A D, or generated in the ſame

time as the line B D, will be of exactly the ſame magnitude on the preſent ſuppoſition, as when the velocity of P was ſuppoſed to be uniform; for if the line A D be divided into the ſame parts AB, BC, CM, *&c.* as before, the line OT will conſiſt of the ſame parts OK, KP, *&c.* as it did before, though the times in which they are generated will be unequal.

256. According to this idea of the generation of logarithms, the magnitude of the logarithm of a given *ratio* depends, as before, on the proportion of the initial velocities of the points P and M at their beginning to move from O and B; or ſuppoſing the velocity of M in B is the ſame in all caſes, and that only the point P varies its initial velocity, the logarithm of a given *ratio* when P ſets out with one velocity will be to the logarithm of the ſame *ratio* when P ſets out with any other velocity, as the former velocity is to the latter. For 'tis evident, that the firſt little portions of the two lines O P generated by the two different velocities of the point P in equal infinitely ſmall times, will be proportional to thoſe velocities, becauſe thoſe velocities may be conſidered as continuing uniform during the times of their deſcription; that is, the logarithms of a given infinitely ſmall *ratio* on the two lines O P deſcribed by the two different initial velocities of P are proportional to thoſe initial velocities. And, by the nature of logarithms, or meaſures of *ratios*, the logarithms of all equal *ratios* in different ſyſtems of logarithms muſt be in the ſame proportion to each other; therefore the logarithms of any given

finite *ratio* in the two ſyſtems of logarithms generated by the point P moving with two different velocities will be to each other in the ſame proportion as the initial velocities of the point P at its ſetting out from O.

257. If the lines O P, A M are generated in the manner deſcribed in the two laſt articles, the velocities of the points P and M when they have deſcribed any two given correſpondent lines as OK and BC, or have arrived at any two given correſpondent points K and C, will be in the ſame proportion to each other as they were before when the velocity of P was ſuppoſed to be an uniform velocity equal to the velocity with which on the preſent ſuppoſition it begins to move.

For ſince CM is deſcribed in the ſame time as K P on this as well as the former ſuppoſition (though this time is not now equal to that in which B C and O K are deſcribed) the uniform velocities with which thoſe lines would be deſcribed in the ſame time as they now actually are deſcribed, will be to each other in the ſame proportion on the one ſuppoſition as on the other, to wit, in the proportion of the lines themſelves; and this, how ſmall ſoever theſe lines be taken. But, if we take them infinitely ſmall, the uniform velocities, with which they would be deſcribed in the ſame times as they now actually are deſcribed will be equal to the velocities with which they begin to be deſcribed, or to the velocities of the points M and P at C and K: therefore the velocities of the points M and P at C and K will be to each other in the ſame proportion on the one ſuppoſition as on the other.

Hence it follows that A E, or the length of the line A M at the inſtant the velocities of P and M are equal to each other, is exactly the ſame when the velocity of P is variable as when it is ſuppoſed to be uniform, provided its firſt velocity in the Point O be the ſame in both caſes. Conſequently it holds in general, whether the velocity of P be uniform or variable, that A E, or the magnitude of A M at the inſtant the velocities of P and M become equal, is equal to the *modulus* of the ſyſtem.

258. 'Tis evident this manner of repreſenting logarithms differs in no other reſpect from that of conſidering them as abſciſſes of the axis of a logarithmic curve than that the continual proportionals correſponding to equal lines or logarithms, are here ſuppoſed to be taken on one and the ſame right line, whereas in the logarithmic curve they are repreſented by different lines placed parallel to one another. For if, at the points O, K, P, T, *t*, &c. (*fig.* 77.) we erect the correſpondent lines AB, AC, AM, AD, A*d*, &c. at right angles to O P, 'tis evident the extremities of theſe lines will be ſituated in a logarithmic curve of which O P is the axis. And hence, that is, on account of this great ſimilitude, and almoſt identicalneſs of theſe two ways of repreſenting logarithms, Dr. *Keill* obſerving, probably, that the contemplation of the logarithmic curve was ſimpler and leſs abſtracted than that of the two ſeparate lines A M, O P, has in explaining *Napier*'s manner of conceiving logarithms

garithms made uſe of a logarithmic curve for that purpoſe, although the logarithmic curve was not known till many years after lord *Napier's* death.

Of the manner in which lord Napier *computed the logarithms of his firſt ſyſtem, commonly call'd* Napier's *logarithms.*

259. As to the manner in which lord *Napier* computed the logarithms of his firſt ſyſtem, it is not mentioned in the book he publiſhed upon this ſubject, entitled *canon mirificus logarithmorum*; but I take it to have been as follows. Since the uniform velocity of the point P was by him ſuppoſed to be equal to the firſt velocity of the point M, it follows that the firſt decrement of the line AM generated in any very ſhort time by the point M at its firſt ſetting out from B, muſt be very nearly equal to the increment of the line OP generated by the point P in the ſame time. If therefore the logarithm of any particular *ratio*, as, for inſtance, the *ratio* of 10 to 1 is to be found, we muſt proceed thus: Between AB and $\frac{1}{10}$ AB, or 1000,000, and 100,000, inſert a mean proportional, which may be done by multiplying AB into $\frac{1}{10}$ AB, or 1000,000 into 100,000, and extracting the ſquare root of the product; and between this mean proportional and AB, or 1000,000, another mean proportional; and ſo on continually, till you come to a number that is exceeded by

AB, or 1000,000, by a quantity leſs than the millionth part of it, or than 1: this quantity, which call q, will be equal to the logarithm of the laſt-found mean proportional, or of the *ratio* of $1000,000 - q$ to 1000,000. Conſequently if q be multiplied by 2, or doubled, as often as there were mean proportionals found in the proceſs made uſe of to obtain it, (or as often as the *ratio* of AB to $\frac{1}{10}$ AB, or of 1000,000 to 100,000, was biſected) the product, which we ſhall find to be 2,302,585, will be the logarithm of the *ratio* of 1000,000 to 100,000, or of 10 to 1. And in the ſame manner, 'tis evident, we may find the logarithm of any other *ratio*.

'Tis evident theſe logarithms are not accurately the ſame that follow from the definition given above of *Napier*'s logarithms, or are not the exact numeral values of the line OP that would ariſe by ſuppoſing AB to be called 1000,000, and the velocity of P to be equal to the firſt velocity of M; becauſe the firſt increment of OP is not, as we have here ſuppoſed it, exactly equal to the contemporary decrement of AM, but ſomewhat greater than that decrement. But they are ſo nearly equal to thoſe logarithms that they will coincide with them for the firſt ſix or ſeven figures; and therefore, where greater exactneſs is not required, may be juſtly conſidered as equal to them. And the exactneſs of this method of computation may be carried to as great a degree as we pleaſe, by increaſing the number of mean proportionals, and conſequently diminiſhing the exceſs of AB, or 1000,000, above the laſt-found mean proportional, *ad infinitum*, (as for inſtance, by diminiſhing that exceſs ſo far, that inſtead of being leſs than $\frac{1}{1000,000} \times AB$, it ſhall be leſs than $\frac{1}{1000,000,000,000} \times AB$) and making the contemporary increment of OP equal to it.

Of the manner in which Mr. Briggs *computed the logarithms of his ſyſtem.*

260. The method by which Mr. *Briggs* computed the logarithms of his ſyſtem is not very different from the foregoing. It is this. He called AB 1, and made the logarithm of 10, or the portion of the line OT generated while AM increaſes to ten times its firſt magnitude, equal to AB, or 1, and then extracted the ſquare-root of 10, and biſected its logarithm, and again extracted the ſquare-root of that ſquare-root and biſected its logarithm, and again extracted the ſquare-root of that laſt ſquare-root and biſected its logarithm, and ſo on continually for fifty-four times ſucceſſively, till he came at laſt to the number 1.000,000,000,000,000,127,819,149,320,032,344,2, the logarithm of which he found, by the continual biſections of the logarithm of 10, to be .000,000,000,000,000,055,511,151,231,257,827,02. This logarithm and number being found, the logarithms of all other numbers may be determined as follows. Extract the ſquare-root of the number whoſe logarithm is to be found, and the

ſquare-root of that ſquare-root, and the ſquare-root of that laſt-found root; and ſo on continually, till you come to a number that is ſo nearly equal to 1, that its exceſs above 1 begins either in the 16th, or ſome lower place of figures below the place of units; and call this exceſs *q*. Then 'tis evident we ſhall have three numbers, to wit, 1, 1 + *q*, and 1.000,000,000,000,000,127 819,149,320,032,344,2, which are all exceeding near to a *ratios* of equality with each other; conſequently the *ratios* of theſe numbers to each other, and the logarithms of thoſe *ratios*, will be proportional to the differences of theſe numbers one from the other, that is, as .000,000, 000,000,000,127,819,149,320,032,344,2, (the exceſs of the laſt of theſe three numbers above the firſt, or 1,) is to *q* (the exceſs of the ſecond of thoſe numbers above the firſt) ſo is .000,000, 000,000,000,055,511,151,231,257,827,02, the logarithm of the *ratio* of the laſt of theſe numbers to the firſt to the logarithm of the *ratio* of 1 + *q* to 1, or of the ſecond number to the firſt. Conſequently the logarithm of the *ratio* of 1 + *q* to 1 will be obtained by this proportion. And when we have got the logarithm of 1 + *q*, we may find the logarithm of the number whereof 1 + *q* is the root, by multiplying the logarithm of the *ratio* of 1 + *q* to 1 by 2 as often as we before performed extractions of the ſquare root to obtain 1 + *q*; for this product will be the logarithm ſought.

261. Theſe are the methods by which Lord *Napier* and Mr. *Briggs* computed their ſyſtems of logarithms. But the practice of them is ſo exceſſively laborious, that mathematicians ſoon ſet about finding eaſier methods of computing

them; and they herein ſo well ſucceded, that by ſome of the infinite ſerieſes invented by modern writers for that purpoſe the labour of computing the logarithm of any given number is rendered almoſt inconſiderable. But of theſe we propoſe to ſay more in another place.

Another manner of conceiving logarithms different from any of the foregoing.

262. Some authors conſider logarithms in the following manner. They conceive a geometrical progreſſion of numbers to be formed whoſe firſt term is unity, and the common *ratio* of the terms exceeding ſmall, or exceeding near a *ratio* of equality; and (as, by reaſon of the ſlow increaſe of the terms, or the ſmall difference between every two contiguous terms, all numbers whatſoever muſt occur in this ſeries, or be equal to ſome of the terms of it, either accurately or very nearly) they conſider all numbers as being equal to ſome or other of the terms of this ſeries, and conſequently as being integral powers of the ſecond term of the ſeries, or of the firſt term after unity; and the indexes of theſe powers, which, 'tis evident, are proportional to their *ratios* to unity, they make the logarithms of thoſe *ratios*. Thus, if the firſt term after unity is 1.000,000,1, it will appear by calculation, that the 23,025,850th term of the ſeries will be (very nearly) equal to 10, and conſequently that 10 is $= \overline{1.000,000,1}|^{23,025,850}$. If therefore 1, or the index of the firſt term after unity, or 1.000,000,1, or $\overline{1.000,000,1}|^{1}$, be made the logarithm of $\overline{1.000,000,1}|^{1}$, (or of its

ratio to 1) 23,025,850, or the index of $1.000,000,1^{23,025,850}$, will be the logarithm of $1.000,000,1^{23,025,850}$, or of 10. And in like manner, the index of that power of the firſt term after unity, to which every other number is equal, will be the logarithm of that number. And thus we have a ſyſtem of *Napier*'s logarithms. If we ſuppoſe the common *ratio* of the terms to be ſuch that there are 10,000,000 terms from 1 to 10, or that 10 is the 10,000,000th term of the ſeries after unity, and the firſt term of the ſeries be called y, 'tis evident 10 will be equal to $y^{10,000,000}$; and in this ſeries it will appear by calculation that 3 is the 4,771,213th term after unity, and is therefore $= y^{4,771,213}$, and 2 is the 3,010,300th term after unity, and therefore is $= y^{3,010,300}$; and every other number will be equal to ſome term or other in the ſeries, and therefore will be equal to ſome integral power, or other, of y, or of the firſt term after unity: and the indexes of theſe powers are proportional to their *ratios* to unity: if therefore we make 1, or the index of y, or y^1, the logarithm of y, or the logarithm of its *ratio* to unity, 'tis evident 10,000,000, or the index of $y^{10,000,000}$, will be the logarithm of $y^{10,000,000}$, or of 10; 4,771,213, or the index of $y^{4,771,213}$, will be the logarithm of $y^{4,771,213}$, or 3; and 3,010,300, or the index of $y^{3,010,300}$, will be the logarithm of $y^{3,010,300}$, or 2. And in like manner the index of that power of y, or of the firſt term after unity, to which every other number is equal, will be the logarithm of that number. And thus we ſhall have a ſyſtem of *Briggs*'s logarithms.

Another conception of logarithms different from the laſt, and all the foregoing ones.

263. Others again conſider *Briggs*'s logarithms in a ſomewhat different manner, not as being the indexes of the integral powers of *y*, to which the numbers whereof they are the logarithms are equal, but as being the indexes of the powers whether integral or fractional, of the number 10 to which the numbers whereof they are the logarithms are equal: for to ſome power of 10, either integral or fractional, 'tis evident every number muſt be equal. They conceive as before, a geometrical progreſſion to be formed, of which unity is the firſt term, and the common *ratio* of whoſe terms is ſuch that the 10,000,000th term of the ſeries after unity is equal to the number 10, and conſequently that the 4,771,213th term after unity is $= 3$, and the 3,010,300th term after unity is $= 2$: but they do not conſider 10 as being the 10,000,000th power of the firſt term after unity, or as being $= y^{10,000,000}$, and 3 as being $= y^{4,771,213}$, and 2 as being $= y^{3,010,300}$, or refer all the terms of the ſeries, to the firſt term *y*, but refer them all to the number 10, or the 10,000,000th term of the ſeries, and conſider *y* as being $= 10^{\frac{1}{10,000,000}}$, or $10^{\cdot 000,000,1}$, and 3 as being $(= y^{4\,771\,213} = \left.\overline{10^{\cdot 000,000,1}}\right)^{4,771,213} = 10^{\cdot 000,000,1 \times 4,771\,213})$ $= 10^{\cdot 477,121,3}$, and 2 as being $(= y^{3,010,300} = \left.\overline{10^{\cdot 000,000,1}}\right)^{3,010,300} = 10^{\cdot 000,000,1 \times 3,010,300}) = 10^{\cdot 301,030,0}$, and in like manner, every other number as being equal to ſome power, either integral or fractional, of the number 10. And the indexes of

theſe powers of 10, which it is evident are every where proportional to their *ratios* to unity, they make the logarithms of thoſe *ratios*; that is, they take 1, or the index of 10, or 10^1, for the logarithm of 10, or the *ratios* of 10 to 1, and conſequently .477,121,3, or the index of $10^{\cdot 477,121,3}$, for the logarithm of $10^{\cdot 477,121,3}$ or 3, and .301,030, or the index of $10^{\cdot 301,030,0}$, for the logarithm of $10^{\cdot 301,030,0}$, and in like manner the index of that power of 10 to which every other number is equal for the logarithm of that number. And thus we have again a ſyſtem of *Briggs*'s logarithms: for, the difference between theſe logarithms, and thoſe found in the laſt article being only in the places of the figures that compoſe them, and not in the figures themſelves, they are, as has been already obſerved, always conſidered as one and the ſame ſyſtem, and are called by the ſame name of *Briggs*'s logarithms.

Recapitulation of the moſt material properties of logarithms that have been mentioned in this diſſertation.

264. But of all thoſe different methods of conceiving logarithms that which was given above at the beginning of this *Diſſertation* ſeems to be the ſimpleſt and the cleareſt, to wit, that they are any quantities that are proportional to, or meaſures of, *ratios*; or, when they are to be conſidered as numbers, that they are the *names*, or *numeral expreſſions* of ſuch quantities: this is the moſt general, clear, and uſeful idea that can, as I apprehend, be formed of logarithms, and the only one that need be remembered. As to what further is moſt material to be remembered

concerning them, it may, I think, be reduced to this, to wit, that there are two geometrical figures more particularly fitted to exhibit theſe quantities than any other figures whatſoever, to wit, the hyperbola and logarithmic curve; and therefore that, when areas are conſidered as logarithms, or meaſures of *ratios*, we ought to have recourſe to the former of thoſe figures, and particularly to that ſimpleſt ſpecies of it, the rectangular, or equilateral, hyperbola; and, when we make uſe of lines for that purpoſe, we ought in general to refer them to the latter figure, or the logarithmic curve; that the parallelogram of an hyperbola is every where of the ſame magnitude in the ſame hyperbola, and the ſubtangent of a logarithmic curve is every where of the ſame magnitude in the ſame logarithmic curve; that the parallelogram of an hyperbola meaſures the ſame *ratio* in that hyperbola as the ſubtangent of a logarithmic curve meaſures in that curve, or that the *ratio* of the ordinates in an hyperbola that bound any aſymptotic area that is equal to the parallelogram of the hyperbola is equal to the *ratio* of any two ordinates to the axis of a logarithmic curve that intercept a portion of the axis equal to the ſubtangent; that the parallelogram of an hyperbola and the ſubtangent of a logarithmic curve are, each of them in its particular ſyſtem of meaſures or logarithms, equal to the fourth proportional which Mr. *Cotes* has called the *modulus* of the ſyſtem; and that the ſyſtem of numeral logarithms, (or names, or numeral expreſſions of the meaſures of *ratios*) known by the name of *Napier*'s logarithms, and ſometimes alſo by thoſe of natural and hyperbolic logarithms, is that which reſults from ſuppoſing the parallelogram of an hyperbola, or the ſubtangent of a logarithmic curve, or, in general, the *modulus* of any ſyſtem of logarithms, to be called 1, or ſome number wherein 1 is the only ſignificant figure, and the ſyſtem of numeral logarithms uſually known by the name of *Briggs*'s logarithms is that which reſults from ſuppoſing the aſymptotic area of an hyperbola intercepted between two ordinates that are to each other in the proportion of 10 to 1, or the abſciſs of the axis of a logarithmic curve that is intercepted between two ordinates that are to each other in the proportion of 10 to 1, or, in general, the meaſure of the *ratio* of 10 to 1 in any ſyſtem of logarithms to be called 1, or ſome number wherein 1 is the only ſignificant figure.

265. *Note*, Though *natural* and *hyperbolic* logarithms are often underſtood to mean the ſame ſyſtem of numeral logarithms, to wit, *Napier*'s, yet ſome authors, I find, diſtinguiſh them one from another, giving only the name of *hyperbolic* to *Napier*'s logarithm, and calling thoſe logarithms *natural* which ariſe by dividing all *Napier*'s, or the hyperbolic, logarithms by 2; thus, according to theſe writers, 2.302,585, &c. is the hyperbolic, or *Napier*'s logarithm of 10, and $\frac{2.302,585, \&c.}{2}$ or 1.151,292, &c. is the natural logarithm of 10. But this diſtinction, I believe, is not a common one.

266. I have now gone through all I propoſed to ſay concerning logarithms, great part of which has been added, not ſo much to explain the nature of theſe quantities (which, I hope, has been done with tolerable perſpicuity in the for-

mer part of this diſcourſe) as to ſatisfy the curioſity of thoſe who might be deſirous of knowing in what manner the firſt inventors, and ſeveral other writers who have conſidered this ſubject, had conceived of them, and of ſeeing the connection of thoſe ſeveral different conceptions of them one with another. With this view all the articles from *Art.* 243, to the preſent article have been compoſed, and they will, it is hoped, contribute to facilitate to young beginners the peruſal of ſome of the moſt curious treatiſes upon this ſubject, particularly Lord *Napier's* *canon mirificus logarithmorum*, Mr. *Maclaurin's* account of logarithms in the 6th chapter of the firſt book of his fluxions, and Dr. *Keill's*, Dr. *Wallis's*, and Dr. *Halley's* treatiſes of logarithms. Thoſe therefore who are not deſirous of knowing all theſe particulars concerning logarithms, which, it muſt be confeſſed, are not neceſſary to the underſtanding them, would do well to give theſe laſt mentioned articles, to wit, from *Art.* 243, to the preſent article, but a ſlight and tranſitory reading.

End of the diſſertation on the nature and uſe of logarithms.

267. We now proceed to demonſtrate the *Corollary* itſelf, which gave riſe to this very long diſſertation, or to ſhew that the variable line that is generated by the motion of a point that begins to move at the ſame inſtant of time that the arc R A (*fig.* 66.) and tangent R P begin to be generated, and continues to move during the whole time of their generation with a velocity that is to the velocity of the point that generates

the tangent R P as P V to P T, or as the radius CA to the ſecant C P, is equal to the logarithm of the *ratio* of the ſum of the ſecant CP and tangent RP to the radius CA in a logarithmic curve, whoſe ſubtangent is the radius C A, or is equal to the portion of the axis of ſuch a logarithmic curve intercepted between two ordinates, whereof the leſſer is equal to the radius CA, and the greater to the ſum of the ſecant C P, and tangent R P. Now this may be ſhewn in the following manner.

Put y for a line that is always equal to the ſum of the tangent R P and exceſs P A of the ſecant C P above the radius C A, and V for the line that is generated by the velocity P V. Then 'tis evident, that the lines y and v will be two variable lines, that both begin to be generated from nothing at the ſame inſtant of time, to wit, at the inſtant the arc R A begins to be generated, and increaſe together continually ever after, and the velocity of y will be to that of v at every inſtant of time during their generation, as PT + TV, the infinitely ſmall increment of y, to PV, the contemporary increment of v; and therefore, on account of the ſimilarity of the triangles PTV, CPR, as CP + RP to CA, or CA + PA + RP to CA, that is, as CA + y is to CA. We have therefore a variable line v that increaſes from o at the ſame time that another variable line CA + y increaſes from a certain given magnitude CA to a greater magnitude CA + y, and the velocity of the increaſe of CA + y is to the contemporary velocity of the line v at every inſtant of time during their generation as the magnitude of the line CA + y

at that inſtant of time is to the given line CA. It follows therefore from *Art.* 205, that the line v is equal to the abſciſs of the axis of a logarithmic curve whoſe ſubtangent is the given line CA that is intercepted between two ordinates, whereof the leſſer is equal to CA, and the greater to CA $+y$, or CP $+$ RP; or the line v is equal to the logarithm of the *ratio* of CP $+$ RP, the ſum of the ſecant and tangent, to CA the radius of the circle, in a logarithmic curve whoſe ſubtangent is the radius CA. QED.

268. It was ſhewn in *Prop.* 23, that the *ratio* of the ſum of the ſecant and tangent of a circular arch to the radius, is equal to any one of the following *ratios*; to wit, the *ratio* of the ſum of the radius and ſine to the coſine; that of the radius to the difference of the ſecant and tangent; that of the coſine to the difference of the radius and ſine; and that of the radius to the tangent of half the complement of the given arch to a quadrant. It follows therefore, that the line v is equal to the logarithm of any one of theſe *ratios*, as well as to that of the *ratio* of the ſum of the ſecant and tangent to the radius, in a logarithmic curve, of which the radius CA is the ſubtangent.

269. It will be of conſiderable uſe in many mathematical inquiries to determine the relations of the velocity of the line v to the velocities of the other lines in a circle as well as to the velocity of the line y, or the ſum of the velocities of the ſecant and tangent: for if this be done, whenever in the ſolution of a mathematical problem, a line occurs that is related to any one

of the lines belonging to a circle in the manner here deſcribed, we ſhall be able to conclude immediately, without any further reaſoning upon it, that it is equal to the logarithm of any one of the *ratios* mentioned in the laſt article in a logarithmic curve of which CA, the radius of the circle, is the ſubtangent. Now 'tis evident, that in *Fig*. 66, we have

PV : PT :: CR : CP :: CD : CA.
PV : TV :: CR : RP :: CD : DA.
PV : BD :: CP*q* : CR x RP :: CA*q* : CD x AD.
PV : A*a* :: CP : CR :: CA : CD.
And PV : *ab* :: CP*q* :: CR*q* ; CA*q* ; CD*q*.

That is, the velocity of the line *v* is to that of the tangent RP as the radius to the ſecant, or coſine to the radius; to that of the ſecant as the radius to the tangent, or coſine to the ſine; to that of the verſed ſine, as the ſquare of the ſecant to the rectangle under the radius and tangent, or as the ſquare of the radius to the rectangle under the ſine and coſine; to that of the arch as the ſecant to the radius or radius to the coſine; and to that of the ſine as the ſquare of the ſecant to the ſquare of the radius, or as the ſquare of the radius to the ſquare of the coſine.

Of the figures of tangents and ſecants.

270. We ſhall now ſhew in what manner the figures of tangents and ſecants may be reduced to certain aſymptotic areas of an hyperbola, and conſequently their areas of an hypertermined by the help of a table of logarithms; for it is evident from the foregoing articles that

every aſymptotic area in an hyperbola may be meaſured by the help of ſuch a table.

271. In the quadrant CAB (*fig.* 78.) of a circle whoſe center is C, and radius C A, take any arcs A E, A G, A N; and draw their tangents AF, AH, A O, and ſecants CF, CH, CO. Then, if the right line *a b* (in *fig.* 79.) be taken equal to the quadrantal arch AB, and the portions of it, *ae*, *ag*, *an* to the arches AE, AG, AN; and at the points *e*, *g*, *n*, be erected the right lines *e*T, *g*V, *np*, at right angles to the line *ab*, and reſpectively equal to the tangents AF, AH, AO, and at every other point in the line *a n* a perpendicular line equal to the correſponding tangent in the circle CAB; and the curve *a*TV*p* be drawn, in which the extremities of all thoſe perpendiculars are ſituated; the figure *a*TV*pna* is called the *figure of tangents*. And, if the right line *a b* (in *fig.* 80.) be taken equal to the quadrantal arch AB, and the portions of it, *ae*, *ag*, *an*, be taken equal to the arches AE, AG, AN; and at the points *e*, *g*, *n*, be erected the right lines, *e d*, *g α*, *n δ*, at right angles to the line *a b*, and reſpectively equal to the ſecants CF, CH, CO, and at every other point in the line *a n* a perpendicular equal to the correſponding ſecant in the circle C A B; and the curve D*dαδ* be drawn, in which the extremities of all theſe perpendiculars are ſituated; the figure *a*D*dαδna* is called the *figure of ſecants*.

272. Let δVT (*fig.* 81.) be a rectangular, or equilateral, hyperbola, whoſe center is C, vertex V, and aſymptotes CQ and CK; and let the ſemi-tranſverſe axis CV be equal to the chord of the

quadrant AB of the circle CAB. Then 'tis evident, that if, from the vertex V we draw the ordinate V*a* parallel to the aſymptote CQ, this ordinate will be perpendicular to the other aſymptote CK, and will be equal to the portion of it C*a*, and each of theſe lines V*a*, C*a* in the hyperbola *δ*VT will be equal to the radius CA of the circle CAB, and conſequently the rectangle under them, or the parallelogram of the hyperbola, will be equal to the ſquare of the radius CA.

273. Now the figure of tangents *a*TV*pna*, correſponding to the circular arch AN, is equal to the logarithm of the *ratio* of the radius CA to the coſine of the arch AN meaſured in this hyperbola, or to the aſymptotic area intercepted between any two ordinates in it that are to each other in the proportion of the radius CA to the coſine of the arch AN; as may be thus demonſtrated.

Draw the ſine NS of the circular arch AN, and in the aſymptote C*a* of the hyperbola *δ*Vt take C*s* equal to the coſine CS of the arch AN, and through the point *s* in the aſymptote C*a* draw the ordinate *sδ* meeting the hyperbola in *δ*. Then, ſince, by the property of the aſymptotes, *sδ* is to V*a* as C*a* to C*s*, and C*a* and C*s* are reſpectively equal to the radius CA and coſine CS, it follows that the ordinates *sδ*, V*a* are to each other in the ſame proportion as the radius CA and coſine CS. We are therefore to ſhew that the figure of tangents *a*TV*pna* is equal to the aſymptotic area V*asδ* intercepted between the ordinates *sδ* and V*a*.

In the circular arch AN take the point L infinitely near to N, and draw LR the ſine, AM the tangent, and CM the ſecant of the arch AL. In the line an (*fig.* 79.) take nl equal to the infinitely ſmall arch NL, and draw the ordinate lk; and from k draw the line kq, parallel to ln, meeting np in q. Laſtly, in the aſymptote Ca take sr equal to the infinitely ſmall difference SR of the coſines CS, CR of the contiguous arches AN, AL, and through the point r in that aſymptote draw the ordinate rd meeting the hyperbola in d; and from d draw the line dt, parallel to Ca, meeting $s\delta$ in t. Then will lk be $= AM$, and $rd = \frac{Ca \times Va}{Cr} = \frac{CAq}{CR} =$ CM. Therefore rd is to lk as CM to AM, or as CL to LR. But, by *Art.* 162, ln is to sr (or LN to SR) as CL to LR. Therefore $rd : lk :: ln : sr$, whence $lk \times ln$ is $= rd \times sr$. But becauſe the ordinates $s\delta$, rd in the hyperbola δVT are infinitely near to each other, it follows that the rectangle $rd \times sr$ is equal to the mixtilinear area δsrd; and for the ſame reaſon the rectangle $lk \times ln$ in the figure of tangents $a$TV$pna$ is equal to the mixtilinear area $klnp$. Therefore the mixtilinear area $klnp$ is equal to the mixtilinear area δsrd. And in the ſame manner it may be ſhewn that, if the figure $a$TV$pna$ be divided, by ordinates drawn infinitely near to each other, into an infinite number of ſuch little areas as $klnp$, and the aſymptotic area δsaV be divided into the ſame number of ſmall areas, ſuch as δsrd, correſponding to thoſe of the figure $a$TV$pna$, every little area in the one figure will be equal to the correſponding little area in the other figure. Conſequently the whole figure a T V p n a is equal to the whole figure δ s a V. QED.

274. The figure of ſecants a D α δ n a, correſponding to the circular arch AN, is equal to the logarithm of the *ratio* of the ſum of the radius CA and ſine NS to the coſine CS meaſured in this hyperbola, or to the aſymptotic area intercepted between any two ordinates in it that are to each other in the proportion of CA + NS to CS; as may be thus demonſtrated.

Take a N (in *fig.* 81.) equal to the ſine NS, and thro' N draw the ordinate Nn meeting the hyperbola in n; and, by the property of the aſymptotes, $s\delta$ will be to Nn as CN, or Ca + aN, to Cs, and conſequently as CA + NS to CS. We are therefore to ſhew, that the figure of ſecants a D α δ n a is equal to the aſymptotic area δ s N n intercepted between the ordinates s δ and N n.

In the right line a n (*fig.* 80.) take n l equal to the infinitely ſmall arch L N, and draw the ordinate $l\gamma$; and from γ draw the line γp, parallel to l n, meeting n δ in p. From L in the circular arch A B draw the line L P, parallel to C A, meeting N S in P. And laſtly, in the aſymptote C N (*fig.* 81.) take N λ equal to N P the infinitely ſmall difference of the ſines N S, L R, and draw the ordinate λ l; and from l draw the line l m, parallel to λ N, meeting N n (produced) in m. Then, by the property of the aſymptotes, we ſhall have Cλ : Ca :: Va : λl, and conſequently $\lambda l = \frac{Ca \times Va}{C\lambda} = \frac{Ca \times Va}{Ca + a\lambda} = \frac{CAq}{CA + LR}$; and, by *Art.* 164, C A : C S ::

$LN : PN$, and conſequently PN, or λN, is $= \frac{LN \times CS}{CA}$. Therefore $\lambda l \times \lambda n$ is $=$ $\frac{CAq \times CS \times LN}{\overline{CA+LR} \times CA} = \frac{CS}{CA+LR} \times CA \times LN =$ $\frac{CA}{CM+AM} \times CA \times LN = \frac{CAq}{CM+AM} \times LN =$ $\overline{CM-AM} \times LN = CM \times LN - AM \times LN$. Conſequently $CM \times LN$, or its equal the rectangle $l\gamma \times ln$ in the figure of ſecants $aD\alpha\delta na$, is $= AM \times LN + \lambda l \times \lambda n$. But $AM \times LN$ is equal to the rectangle $lk \times ln$ in the figure of tangents $aTVpna$, or, by *Art.* 273, to the rectangle $rd \times sr$ in the hyperbola δVT. Therefore the rectangle $l\gamma \times ln$ in the figure of ſecants $aD\alpha\delta na$ is equal to the ſum of the two rectangles $rd \times sr$ and $\lambda l \times \lambda N$ in the hyperbola δVT. But, becauſe the ordinates $l\gamma$, nd (in *fig.* 80.) are infinitely near to each other, the rectangle $l\gamma \times ln$ will be equal to the mixtilinear area $\gamma ln\delta$; and for the ſame reaſon the rectangles $rd \times sr$ and $\lambda l \times \lambda N$ will be equal to the mixtilinear areas δsrd and $l\lambda Nn$. Conſequently the mixtilinear area $\gamma ln\delta$ in the figure of ſecants $aD\alpha\delta na$ is equal to the ſum of the two mixtilinear areas $dsrd$ and $l\lambda Nn$ in the hyperbola δVT. And in the ſame manner it may be ſhewn, that if the figure $aD\alpha\delta na$ be divided, by ordinates drawn infinitely near to each other, into an infinite number of ſuch little areas as $\gamma ln\delta$, and each of the aſymptotic areas δsaV and $VaNn$ be divided into the ſame number of ſmall areas, ſuch as δsrd and $l\lambda Nn$, correſponding to thoſe of the figure $aD\alpha\delta na$, every area in the figure *a*D*αδna*, will be equal to the ſum of the two correſponding little portions of the aſymptotic areas *δ s a* V and *ι* λ N *n*. Conſequently the whole figure of ſecants *a*D*αδna* is equal to the ſum of the two aſymptotic areas *δsa* V and V *a* N *n*, or to the aſymptotic area *δs* N *n*. QED.

275. It follows from theſe two demonſtrations, that the figure of ſecants *a*D*αδna* is equal to the ſum of the correſponding figure of tangents *a*TV*pna* and the aſymptotic area *δsa* V.

276. And, becauſe, by *Art.* 213, the aſymptotic areas correſponding to the ſame *ratio* in different hyperbolas are always in the ſame proportion to each other as the parallelograms of thoſe hyperbolas, however the ſpecies of thoſe hyperbolas, or the proportion of their axes, may differ one from the other, it follows that the figure of tangents *a*TV*pna* correſponding to any given circular arch AN is equal to the logarithm of the *ratio* of the radius of the circle to the coſine of that arch in any hyperbola whatſoever (as well as a rectangular one) the parallelogram of which is equal to the ſquare of the radius of the circle; and the figure of ſecants *a*D*αδna* correſponding to any given circular arch AN is equal to the logarithm of the *ratio* of the ſum of the radius of the circle and the ſine of that arch to its coſine, or to the ſum of the logarithms of the *ratios*, of the radius of the circle to the coſine of that arch, and of the *ſum* of the radius and the ſine of that arch to the radius, or laſtly, to the ſum of, the figure of tangents *a*TV*pna* correſponding to that arch, and the logarithm of the *ratio* of the ſum of the radius of the circle

and the ſine of that arch to the radius, in any hyperbola whatſoever (as well as a rectangular one) the parallelogram of which is equal to the ſquare of the radius of the circle.

277. The figures of tangents and ſecants are evidently infinite in length, or the points *n* (in *fig.* 79 and 80.) may be taken ſo near to the points *b* that the diſtances of the curves *a*TV*py* and *a*D*αδy* from their baſes at thoſe points, or the lengths of the ordinates *np*, *nδ*, ſhall be a greater than any aſſigned length whatſoever. For, as the arch AN may be taken ſo nearly equal to the quadrant AB, that its tangent AO and ſecant CO ſhall each of them exceed any aſſigned line whatſoever, and the whole baſes *a b* are equal to the quadrant A B, the abſciſſes *a n* to the arch AN, and the ordinates *np*, *n δ* to the tangent A O and ſecant C O, it follows that the abſciſſes *a n* may be taken ſo nearly equal to the whole baſes *a b* that the ordinates *np*, *n δ* ſhall be greater than any aſſigned line whatſoever. QED.

278. Hence it is evident, that if, through the extreme points *b* of the baſes *a b* we draw the lines *b x* perpendicular to them, the curves *a*T V *p y* and D *α δ y* will approach continually to theſe perpendiculars, and may be made to come ſo near them, that their diſtance from them ſhall be leſs than any aſſigned line whatſoever, but will never actually meet them; or, in other words, the perpendiculars *b x* are *aſymptotes* to the curves *a*TV*py* and D*αδy*.

279. Further, the figures of tangents and ſecants are infinite in magnitude as well as in length, or the abſciſſes *a n* (in *fig.* 79 and 80.)

may be taken ſo nearly equal to the whole baſes *a b* that the areas *a*TV*pna* and *a*D*αδna* ſhall each of them be greater than any aſſigned area how great ſoever. For ſince theſe areas are equal to the logarithms of the *ratios* of the radius C A to the coſine of the arch A N, and of the ſum of the radius and the ſine of that arch to its coſine, in any hyperbola, as *δ*VT, the parallelogram of which is equal to the ſquare of the radius C A; and the arch A N may be taken ſo nearly equal to the quadrant A B that the *ratio* of the radius to its coſine, and, *a fortiori*, the *ratio* of the ſum of the radius and its ſine to its coſine, ſhall be greater than any aſſigned *ratio* whatſoever, and conſequently that the logarithms of theſe *ratios*, or the aſymptotic areas in the hyperbola *δ*VT that meaſure them, ſhall be greater than the logarithm of any aſſigned *ratio*, or than the aſymptotic area in that hyperbola that meaſures any aſſigned *ratio*; and the lines *a b* are equal to the quadrant A B, and the abſciſſes *a n* to the arch A N; it follows, that the abſciſſes *a n* may be taken ſo nearly equal to the lines *a b* that the areas *a*TV*pna* and *a*D*αδna* ſhall each of them be greater than the aſymptotic area that meaſures any aſſigned *ratio* how great ſoever in the hyperbola *δ*VT, and conſequently than any aſſigned area how great ſoever. QED.

280. The curves *a*TV*py* and D*αδy* of the figures of tangents; and ſecants *a*TV*pna* and *a*D*αδna* are convex towards their baſes *a b*; as may be thus demonſtrated. Let the arc E G be ſuppoſed to be equal to G N, and conſequently

the portions $e\,g$ of the abſciſſes $a\,n$ (in *fig.* 79 and 80) to be equal to the portions $g\,n$. Draw the ordinates eT, gV, np, and ed, $g\alpha$, $n\delta$; and the chords Tp, $d\delta$, cutting the middle ordinates gV, $g\,\alpha$ (produced, if need be) in b and m. From V draw $V\Pi$, parallel to $a\,b$, meeting $n\,p$ in Π, and from T draw $T\pi$, parallel to $a\,b$, meeting gV in t and $n\,p$ in π; and in like manner from α draw $\alpha\,\theta$ parallel to $a\,b$, meeting $n\,\delta$ in θ, and from d draw $d\phi$, parallel to $a\,b$, meeting $g\,\alpha$ in s, and $n\,\delta$ in ϕ.

Then it is evident, in the firſt place, that we ſhall have $t\,V$ ($=gV-gt=gV-eT=AH-AF$) $=FH$, and Πp ($=np-n\Pi=np-gV=AO-AH$) $=HO$. But, becauſe the arcs EG, GN are equal, 'tis plain HO will be greater than FH. Therefore Πp is greater than tV, and conſequently tV is leſs than half the ſum of Πp and tV, or than half πp. But from the ſimilarity of the triangles btT, $p\pi T$, it follows that $t\,b$ is to $\pi\,p$ as $T\,t$ to $T\pi$, or $e\,g$ to $e\,n$; and $e\,g$ is half of $e\,n$; therefore $t\,b$ is half of πp, and conſequently is greater than $t\,V$. Therefore the arch $T\,p$ falls below its chord, or is nearer than its chord to the baſe $a\,b$, and therefore is convex towards that baſe; that is, the curve of the figure of tangents is convex towards its baſe. QED.

Secondly, 'Tis plain that in *Fig.* 80, we ſhall have $s\,\alpha$ ($=g\,\alpha-g\,s=g\,\alpha-e\,d$) $=CH-CF$, and $\theta\delta$ ($=n\delta-n\theta=n\delta-g\,\alpha$) $=CO-CH$. But, becauſe the arcs EG, GN are equal, it follows, from *Prop.* 24, *Corol.* 3 and 4, that CO — CH is greater than CH — CF. Therefore $\theta\delta$ is greater than $s\alpha$, and conſequently $s\,\alpha$

is leſs than half the ſum of $\theta\delta$ and $s\alpha$, or than half $\phi\delta$. But from the ſimilarity of the triangles msd, $\delta\phi d$, it follows that sm is to $\phi\delta$ as ds to $d\phi$, or eg to ez; and eg is half of ez; therefore sm is half of $\phi\delta$, and conſequently is greater than $s\alpha$. Therefore the arch $d\delta$ falls below its chord, or is nearer than its chord to the baſe ab, and therefore is convex towards that baſe; that is, the curve of the figure of ſecants is convex towards its baſe. QED.

281. If, inſtead of the arch A N, we make its tangent A O the baſe upon which we erect the ſecants perpendicularly, the curve that will be thereby formed, or in which the extremities of all the ſecants will be ſituated, is a rectangular hyperbola, whoſe axis is equal to the diameter of the circle CAB. Let the right line ao (*fig.* 82.) be equal to the tangent A O, and the ſeveral portions of it af, ah, ak, am to the tangents AF, AH, AK, AM; and at the points a, f, h, k, m, o, erect the perpendiculars aD, fb, hc, kd, me, og, reſpectively equal to the radius CA and the ſecants CF, CH, CK, CM, CO: the curve D$bcdegy$, in which the extremities of all theſe perpendiculars are ſituated, is a rectangular, or equilateral hyperbola, whoſe axis is equal to 2 C A, or the diameter of the circle CAB. For 'tis evident, from the generation of this curve, that the ſquare of every perpendicular to the baſe ao is equal to the ſum of the ſquares of the correſponding abſciſs of that baſe and of the line aD; thus, kdq is ($= \mathrm{CK}q = \mathrm{AK}q + \mathrm{CA}q$) $= akq + a\mathrm{D}q$; which is the property of an equilateral hyperbola whoſe ver-

tex is D and center *a*, or whoſe ſemi-axis is *a* D, and therefore is equal to the radius CA, and conſequently its whole axis to 2CA, or the diameter of the circle CAB. QED.

282. This manner of generating an equilateral hyperbola, by deriving it from the circle, we have here mentioned, partly as a previous ſtep to the demonſtration which we ſhall juſt now give of a very curious and remarkable analogy between the external area *aDgo* of an equilateral hyperbola and the curve of a parabola, and partly as an inſtance to ſhew, that the properties of the lines belonging to a circle, may be of uſe in determining thoſe of other figures, with which one would not at firſt ſight imagine it was ſo nearly connected: indeed, not only the equilateral hyperbola, but alſo ellipſes and parabolas may be derived from the circle by very eaſy and obvious methods; for, from the equality between the ſquare of the chord of a circular arch, and the rectangle under the verſed ſine of that arch, and the diameter of the circle, it follows that, if the chords of a ſet of circular arches be placed at right angles to the diameter of the circle at the extremities of the reſpective verſed ſines of thoſe arches, or in thoſe points of the diameter where the ſines of thoſe arches meet it, the extremities of thoſe chords will be ſituated in the curve of a parabola, whoſe principal parameter is equal to the diameter of the circle; and 'tis well known that, if a ſet of ſines, or of lines drawn perpendicular to a given diameter of a circle, be all either increaſed or diminiſhed in the ſame proportion, the curve

that will be thereby formed, or in which the extremities of theſe lines will be ſituated, will be an ellipſe. And in like manner, many other uſeful curves may be derived from the circle, to the ready underſtanding of which, the knowledge of the properties of the lines belonging to a circle will be found extremely uſeful.

283. Another thing to be obſerved concerning the foregoing manner of generating the hyperbola D*dy*, is that it is as good and as eaſy a practical method as can be deſired, of actually deſcribing a rectangular hyperbola, by finding as many of its points D, *b*, *c*, *d*, &c. as we pleaſe.

A remarkable analogy between the external area of a rectangular hyperbola and the curve of a parabola.

284. The analogy juſt now ſpoken of between the external area *a*D*go* of the rectangular hyperbola D*dy* and the curve of a parabola is this. If in the parabola GVH (*fig.* 83.) whoſe vertex is V and focus F, and whoſe parameter is equal to *a*D the ſemi-axis of the hyperbola D*dy*, we take the ordinate GH to the axis equal to the abſciſs *a o* of the ſecond axis of that hyperbola, and cutting the axis of the parabola in E, the parabolic arch G V will be to the ſemi-ordinate GE, and conſequently the parabolic arch GVH to the whole ordinate GH, in the ſame proportion as the hyperbolic area *a*D*go* to the rectangle under the abſciſs *a o* and ſemi-axis *a* D; or, if

4VF is = *a*D, and GH is = *a o*, we ſhall have GVH : GH :: *a*D*go* : *a* D × *a o*.

Conceive the parabolic ſegment GVHG to be divided into an infinite number of infinitely ſmall parts, (whether equal or unequal, is of no conſequence) by the ſeveral ordinates to the axis, *tv*, *rs*, *rs*, *rs*, &c. of which let *t v* be the ordinate next above the baſe GH. Take the portion *a m* of the line *a o* equal to the ordinate *t v*, and the following portions of *a o*, to wit, *az*, *az*, *az*, &c. equal to the following ordinates *rs*, *rs*, *rs*, &c. and through the extremities *m*, *z*, *z*, *z*, &c. of theſe portions of *a o* draw the ordinates *m e*, *zp*, *zp*, *zp*, &c. meeting the hyperbola in *e*, *p*, *p*, *p*, &c. From D, the vertex of the hyperbola D*ay*, draw DP parallel to *a o*, meeting *o g* in P; from *t*, the extremity of the ordinate *t v*, draw *t l*, parallel to the axis EV, meeting the ordinate GH in *l*, and from the extremities *r*, *r*, *r*, &c. of all the other ordinates *rs*, *rs*, *rs*, &c. draw lines parallel to the axis EV, meeting the next lower ordinates in the points *b*, *b*, *b*, &c. and conceive theſe lines to be produced till they meet the loweſt ordinate GH in the points *n*, *n*, *n*, &c. 'Tis evident, that by this means the hyperbolic area *a*D*go*, and the rectangle *a*DP*o* will each of them be divided into the ſame number of infinitely ſmall parts by the ordinates *m e*, *zp*, *zp*, *zp*, &c. as the parabolic arch GV is divided into by the ordinates *tv*, *rs*, *rs*, *rs*, &c. and the ſemi-ordinates GE by the lines *tl*, *rn*, *rn*, *rn*, &c. If therefore we can ſhew that the infinitely ſmall mixtilinear area *e g o m* is to the infinitely ſmall rectangle *m* P in the ſame proportion as the infinitely ſmall portion G*t* of the parabolic arch GV to the infinitely ſmall portion G*l* of the ſemi-ordinate G E, and that every one of the following mixtilinear areas *p e m z*, *p p z z*, &c. is to the correſpondent portion of the rectangle *a*DP*o* in the ſame proportion as the correſponding portion *tr*, or *rr*, of the parabolic arch G V is (to the difference *t b* or *r b* of the halves of the two contiguous ordinates *t v* and *r s*, or *r s* and *r s*, between which that portion of the arch GV is intercepted, or) to the correſponding portion *l n*, or *n n*, of the ſemi-ordinate G E, it will follow that the ſum of all the mixtilinear areas *e g o m*, *p e m z*, *p p z z*, &c. will be to the ſum of all the correſpondent portions of the rectangle *a*DP*o*, in the ſame proportion as the ſum of all the infinitely ſmall portions G*t*, *tr*, *rr*, &c. of the parabolic arch GV, to the ſum of all the portions G*l*, *ln*, *nn*, &c. of the ſemi-ordinate GE; that is, the whole hyperbolic area *a*D*go* will be to the whole rectangle *a*DP*o* in the ſame proportion as the whole parabolic arch GV to the whole ſemi-ordinate G E, and conſequently as the parabolic arch GVH to the whole ordinate GH.

Draw GK touching the parabola GVH in G and meeting its axis in K, and from the extremity *g* of the ordinate *o g* in the hyperbola D *d y* draw *g q* parallel to *a o*, meeting *m e* (produced) in *q*. This done, the propoſition may be demonſtrated as follows.

Becauſe the arch G*t* is infinitely ſmall, it may be conſidered as coinciding with the tangent GK; conſequently the triangles G *t l* and GKE will be ſimilar, and G*t* will be to G *l* as GK to

GE. But, becauſe the parameter of this parabola is = aD = the radius CA of the circle CAB, it follows, that CA : GE :: GE : EV, and conſequently CA : 2 GE :: GE : 2 EV, and therefore (becauſe the tangent AO is = ao = GH = 2 GE, and, by *Simſon*'s Con. Sect. *lib.* 1. *prop.* 10, the ſubtangent EK is = 2 EV) CA : AO :: GE : EK. Therefore (by *El.* 6. 6.) the triangles CAO, and GEK are ſimilar, and conſequently GK : EK :: CO : CA. Therefore Gt is to Gl :: CO : CA :: og : aD :: og : oP :: $og \times om$: oP $\times om$; that is, the infinitely ſmall rectangle $ogqm$ is to infinitely ſmall rectangle mP as the infinitely ſmall portion Gt of the parabolic arch GV correſponding to the rectangle $ogqm$ to the difference Gl of the halves of the two contiguous ordinates GH and tv, between which the ſmall arch Gt is intercepted. But becauſe the ordinates me, og are infinitely near each other, the mixtilinear area $egom$ will be equal to the rectangle $ogmq$; therefore the mixtilinear area $egom$ is to the rectangle mP as Gt to Gl. And in the ſame manner it may be ſhewn that every following mixtilinear area $pemz$, $ppzz$, &c. in the hyperbola Ddy is to the correſponding portion of the rectangle $a$DP$o$, as the portion tr, or rr, of the parabolic arch GV, correſponding to that mixtilinear area, to the difference of the halves of the two contiguous ordinates, tv, and rs, or rs and rs, between which that portion of the arch GV is intercepted, or to the portion ln, or nn, of the ſemi-ordinate GE correſponding to that portion of the arch GV. Therefore the whole hyperbolic area

$aDgo$ is to the whole rectangle $aDPo$ as the whole parabolic arch GV to the whole ſemi-ordinate GE, and conſequently as the parabolic arch GVH to the whole ordinate GH. QED.

285. From the foregoing article it follows, that the hyperbolic area $aDgo$ is equal to a rectangle whoſe baſe is equal to the parabolic arch GVH and altitude to the parameter of that parabola, or to the ſemi-axis aD. For $aDgo : ao \times aD :: GVH : GH :: GVH : ao :: GVH \times aD : ao \times aD$. Therefore $aDgo$ is $= GVH \times aD$.

Mr. Cotes's *conſtruction of a parabolic arch.*

286. Having thus demonſtrated this analogy between the hyperbolic area $aDgo$ and the parabolic arch GVH, it will not be difficult to deduce from it the elegant conſtruction which Mr. *Cotes* has given for finding the length of a parabolic arch in his *Harmonia menſurarum*, p.22. This conſtruction is as follows. Biſect the ſemi-ordinate GE in L, and draw the line LV, and produce it to M, ſo that LM ſhall be the logarithm of the *ratio* of the ſum of the lines LV and VE to the line LE in a ſyſtem of logarithms whoſe *modulus* is VF, or a fourth part of the parameter of the parabola GVH, or ſhall be equal to ſuch a portion of the axis of a logarithmic curve, whoſe ſubtangent is equal to VF, as is intercepted between any two ordinates that are to each other in the proportion of LV + VE to LE; the right line VM will be equal to the parabolic arch GV.

A demonſtration of this conſtruction.

In order to demonſtrate this conſtruction, draw the aſymptote aR of the hyperbola Ddy, cutting the ordinate og in α, and from the points D and g draw the lines DN, gR perpendicular to this aſymptote, and meeting it in N and R.

Then 'tis evident in the firſt place, that the right angle Dao is biſected by the aſymptote aR, and conſequently, that the angles Rao, RaD, aDN, $a\alpha o$, $R\alpha g$, and $Rg\alpha$, are each of them equal to half a right angle. Therefore $o\alpha$ is $=$ oa, $\alpha R = gR$, and $DN = aN$. Therefore aN^q is $= \frac{aD^q}{2}$, and $\alpha R^q = \frac{\alpha g^q}{2}$; and the triangle aND is $= \frac{aN^q}{2} = \frac{aD^q}{4}$, the triangle αRg $= \frac{\alpha R}{2} = \frac{\alpha g^q}{4}$, and the triangle $ao\alpha = \frac{ao^q}{2}$. Further the parallelogram of this hyperbola is $=$ $DN \times aN = aN^q = \frac{aD^q}{2}$; and, becauſe the triangles DaN, $g\alpha R$ are ſimilar, and conſequently aD is to αg as DN to gR, the aſymptotic area $DNRg$, which meaſures the *ratio* of DN to gR, will alſo be the meaſure of the *ratio* of aD to αg.

2*dly*, Becauſe GL is taken = LE, and VK is, by the nature of the parabola = VE, the triangles GKE and LVE are ſimilar; but it was ſhewn in the preceding demonſtration, that the triangles GKE and COA are ſimilar; therefore the triangles LVE and COA are ſimilar, and LE is to LV as CA to CO, and conſequently

CO is $= \frac{CA \times LV}{LE}$. Alſo the *ratio* of LV + VE to LE will be equal to the *ratio* of CO + AO to CA.

Therefore GVH × CA is

$= GVH \times aD$

$=$ hyperbolic area $aDgo$

$=$ triangle $a\alpha o$ + triangle aDN + the mixtilinear area $DN\alpha g$

$=$ triangle $a\alpha o$ + triangle aDN + aſymptotic area $DNRg$ — triangle αRg

$$= \frac{ao^q}{2} + \frac{aD^q}{4} + \log. \frac{aD}{\alpha g} - \frac{\alpha g^q}{4}$$

$$= \frac{ao^q}{2} + \frac{aD^q}{4} + \log. \frac{aD}{og - o\alpha} - \frac{\overline{og - o\alpha}^{q}}{4}$$

$$= \frac{ao^q}{2} + \frac{aD^q}{4} + \log. \frac{aD}{og - ao} - \frac{\overline{og - ao}^{q}}{4}$$

$$= \frac{ao^q}{2} + \frac{aD^q}{4} + \log. \frac{aD}{og - ao} \frac{-og^q + 2og \times ao - ao^q}{4}$$

$$= \frac{ao^q}{2} - \frac{ao^q}{4} + \frac{aD^q}{4} - \frac{og^q}{4} + \log. \frac{aD}{og - ao} + \frac{2og \times ao}{4}$$

$$= \frac{ao^q}{4} + \frac{aD^q}{4} - \frac{og^q}{4} + \log. \frac{aD}{og - oa} + \frac{og \times ao}{2}$$

$$= \frac{AO^q}{4} + \frac{CA^q}{4} - \frac{CO^q}{4} + \log. \frac{CA}{CO - AO} + \frac{CO \times AO}{2}$$

$$= \log. \frac{CA}{CO - AO} + \frac{CO \times AO}{2}$$

$$= \log. \frac{CO + AO}{CA} + \frac{CO \times AO}{2}$$

$$= \log. \frac{LV + VE}{LE} + \frac{CO \times GH}{2}$$

= log.

$$= \log. \frac{LV+VE}{LE} + CO \times 2\,LE$$

$$= \log. \frac{LV+VE}{LE} + \frac{CA \times LV}{LE} \times 2\,LE$$

$= \log. \frac{LV+VE}{LE} + 2CA \times LV$, and conſequently $GV \times CA$ is $= CA \times LV + \frac{1}{2} \log. \frac{LV+VE}{LE}$; that is, the rectangle under the arch GV and CA, the parameter of the parabola, is equal to the ſum of the rectangle under the right line LV and parameter CA and half the aſymptotic area that meaſures the *ratio* of $LV + VE$ to LE in the hyperbola Ddy whoſe parallelogram is $= \frac{aD^q}{2}$ or $\frac{CA^q}{2}$. But the half of the aſymptotic area that meaſures this *ratio* in the hyperbola Ddy whoſe parallelogram is $\frac{CA^q}{2}$ is equal to the whole aſymptotic area that meaſures the ſame *ratio* in an hyperbola whoſe parallelogram is but half of the parallelogram of this hyperbola, or is equal to $\frac{CA^q}{4}$. Therefore $GV \times CA$ is $= CA \times LV +$ the aſymptotic area that meaſures the *ratio* of $LV + VE$ to LE in an hyperbola whoſe parallelogram is $\frac{CA^q}{4}$. But (by *Art.* 216) this aſymptotic area is equal to the rectangle whoſe baſe is ſuch a portion of the axis of a logarithmic curve whoſe ſubtangent is $\frac{CA}{4}$, or VF, as meaſures the ſame *ratio* of $LV + VE$ to LE that in that curve, and whoſe altitude is equal to CA, that is, to the rectangle LM x CA. Therefore GV x CA is = LV x CA + LM x CA, and conſequently GV is = LV + LM = VM. QED.

Of the radius of curvature of a logarithmic curve.

287. We ſhall conclude this *Corollary* with a determination of the radius of curvature of the logarithmic curve at any point aſſigned, and of the point where this radius is leaſt, or the curvature of this curve is greateſt; for 'tis eaſy to perceive that the curvature of this line decreaſes *ad infinitum* on both ſides of any given point in it, and conſequently that there muſt be ſome one point in it at which the curvature is greater than at any other point in it whatſoever. But firſt we muſt obſerve, that in the following determination of the radius of curvature it is taken for granted that, the limit of the interſections of two contiguous perpendiculars to the tangents of a curve; or the point to which, as the diſtance of theſe perpendiculars, or the arch intercepted between them, is continually diminiſhed, their interſection continually approaches, and to which it may be made to come nearer than within any aſſigned diſtance; or, in the language of infiniteſimals, the ultimate interſection of theſe perpendiculars when the arch intercepted between them is infinitely ſmall; I ſay, it is here taken for granted, that this point is the center of the circle, which has the ſame curvature with the curve in the point where the upper extremities of the two perpendiculars coincide;

incide; or of the circle which has the ſame tangent with the curve in that point, and touches it ſo cloſely that no other circle can paſs between that circle and the curve. Thus let B F (*Fig.* 84 be a ſmall arch of any curve, BG a line perpendicular to the tangent at B, and G the point to which, if we ſuppoſe the point F to move towards B, the interſection of a perpendicular to the tangent at F will continually approach as to its limit; it is here ſuppoſed that the point G will be the center, and the line BG the radius of the circle that has the ſame curvature with the curve BF in the point B, or between which and the curve B F no other circle can be drawn. Whoever is deſirous of ſeeing an accurate demonſtration of this may find one in the 402^{d} article of Mr. *Maclaurin*'s fluxions; for it is there demonſtrated, that the tangent to the evolute curve is the radius of a circle of equal curvature with the curve of evolution at the point where it meets that curve; and from the nature of evolute lines 'tis evident that the other extremity of this tangent, or radius of curvature, which is ſituated in the evolute line, is the limit of the interſection of two contiguous tangents to the evolute, or of two contiguous perpendiculars to the curve of evolution. This article of Mr. *Maclaurin*'s fluxions contains a definition of evolute curves and curves of evolution, as well as a demonſtration of the property of them here mentioned, and is totally independent of all that goes before, and therefore may be read and underſtood by itſelf without conſulting either any other book, or any other part of that book.

As to a demonſtration of this property of the limit of the interſections of two contiguous perpendiculars (or of the co-incidence of this limit with the center of curvature) independent of the properties of evolute lines, and derived merely from the conſideration of tangents and perpendiculars, it is what I have not been able to meet with in any of the authors I have ſeen; and, I believe it would be pretty difficult to make. This being premiſed, the radius of curvature in the logarithmic curve may be found as follows.

288. Let ABT (*Fig.* 84.) be a logarithmic curve whoſe axis is SM; B and F any two points in it infinitely near to each other; BK and FL perpendiculars to the axis SM, drawn through thoſe points, and meeting the axis in K and L; BH and F*b* tangents to the curve in thoſe points, meeting the axis in H and *b*; and BG and FG perpendiculars to the tangents BH and F*b*, meeting each other in G, and the axis in M and N. Laſtly, draw BO, parallel to the axis, meeting the ordinate LF (produced) in P, and the perpendicular FG in O.

Then 'tis evident the triangles GBO, GMN will be ſimilar, and conſequently GM will be to GB as MN to BO; and, *dividendo*, BM : BG :: MN — BO : BO. If therefore we can find the magnitude of BM, and the proportion of MN—BO to BO, or, which amounts to the ſame thing, the proportion of MN to BO, the magnitude of BG will be thereby determined.

Now, becauſe the arch BF is infinitely ſmall, and conſequently may be conſidered as coinciding with the tangent BH, it follows that the triangles BPF, BOF will be rectilinear triangles ſimilar to each other, and to the triangles BHK, BKM, BHM. And, 'tis evident likewiſe, that the triangles FL*b*, FLN, and F*b*N are ſimilar to each other.

Therefore in the *firſt* place we have KH : BH :: BK : BM, whence BM is $= \frac{BH \times BK}{KH}$.

2*dly*, We have KH : BK :: BK : KM; and L*b* : LF, or BK — PF :: LF, or BK — PF : LN; and BP, or KL : PF :: KH : BK. Therefore KM is $= \frac{BK^q}{KH}$; LN is $=$ $\frac{\overline{BK - PF}^q}{Lb}$ $=$ (becauſe by the nature of the curve the ſubtangents K H, L*b* are equal) $\frac{\overline{BK - PF}^q}{KH} = \frac{BK^q - 2BK \times PF + PF^q}{KH}$; and KL is $= \frac{KH \times PF}{BK}$. Therefore MN ($=$ KM — KN $=$ KM — LN $+$ KL) is $= \frac{BK^q}{KH} -$

$$\frac{BK^q + 2BK \times PF - PF^q}{KH} + \frac{KH \times PF}{BK} =$$

$$\frac{2BK \times PF - PF^q}{KH} + \frac{KH \times PF}{BK}$$

$$\frac{2BK^q \times PF - BK \times PF^q + KH^q \times PF}{BK \times KH} =$$

(becauſe PF is infinitely ſmaller than BK and KH, and conſequently BK $\times$ PF^q than 2BK^q $\times$ PF and KH^q $\times$ PF) $\frac{2BK^q \times PF + KH^q \times PF}{BK \times KH}$.

Therefore 3*dly*, MN is to KL as $\frac{2BK^q \times PF + KH^q \times PF}{BK \times KH}$ to $\frac{KH \times PF}{BK}$, or $\frac{KH^q \times PF}{BK \times KH}$, and therefore as $2\,BK^q + KH^q$ to KH^q, or as $BK^q + BH^q$ to KH^q. But $KL : BO :: BP : BO :: BP^q : BF^q :: KH^q : BH^q$. Therefore, *ex æquo*, MN is to $BO :: BK^q + BH^q : BH^q$, and, *dividendo*, $MN - BO : BO :: BK^q : BH$. But $BM : BG :: MN - BO : BO$; therefore $BM : BG :: BK^q : BH^q$. And it was before ſhewn that BM is $= \frac{BH \times BK}{KH}$; therefore $BK^q : BH^q :: \frac{BH \times BK}{KH} : BG$; conſequently BG is $= \frac{BH^q}{BK^q} \times \frac{BH \times BK}{KH} = \frac{BH^3}{BK \times KH}$. QED.

289. Hence we may derive the following very elegant and ſimple conſtruction of the radius of curvature BG at the point B. From H, the point in which the tangent BH cuts the axis SM, draw HR perpendicular to the axis, meeting the perpendicular to the tangent BH in R: in line MR will be equal to the radius of curvature BG. For MR is to $MH :: MH : MB :: BH : BK$; and $MH : KH :: BH^q : KH^q$. Therefore $MR : KH :: BH^3 : BK \times KH^q$ and conſequently $MR = \frac{BH^3 \times KH}{BK \times KH^q} = \frac{BH^3}{BK \times KH} = BG$. QED.

This conſtruction is given by Mr. *John Bernouilli*. [See vol. ii. of his works, p. 383.]

290. Since the radius of curvature BG is $= \frac{BH^3}{BK \times KH} = \frac{BH^3 \times KH}{BK \times KH^q}$, or is in the ſame proportion to the ſubtangent KH as BH^3 to $BK \times KH^q$, it follows that it muſt always be greater than that ſubtangent. For, as BH ſubtends the right angle BKH, 'tis evident it muſt always be greater than either BK or KH, and conſequently BH^3 will always be greater than $BK \times KH^q$.

291. Further, as the ordinate BK increaſes *ad infinitum*, the tangent BH increaſes *ad infinitum* likewiſe; conſequently its proportion to the ſubtangent KH, which is always of a given magnitude, and the proportion of its ſquare to the ſquare of KH, increaſes *ad infinitum*. But its proportion to the ordinate BK evidently approaches to a *ratio* of equality as its limit. Therefore the ſum of the *ratios* of BH^q to KH^q and of BH to BK, or the *ratio* of BH^3 to $BK \times KH^q$, increaſes *ad infinitum*; that is, the *ratio* of the radius of curvature BG to the ſubtangent KH increaſes *ad infinitum* as the ordinate BK increaſes; and conſequently, as the ſubtangent KH is always of a given magnitude, the radius of curvature increaſes *ad infinitum* as the ordinate BK increaſes.

292. Again, as the ordinate BK decreaſes *ad infinitnm*, the tangent BH decreaſes to a certain given magnitude as its limit, to wit, the ſubtangent KH; conſequently its proportion to BK increaſes *ad infinitum*, and its proportion to KH, and conſequently the proportion of its ſquare to the ſquare of KH, approaches continually to a

ratio of equality as its limit. Therefore the ſum of *ratios* of the BH to BK and of BH^q to KH^q, or the *ratio* of BH^3 to $BK \times KH^q$, increaſes *ad infinitum*; that is, the *ratio* of the radius of curvature BG to the ſubtangent KH increaſes *ad infinitum*; and conſequently, as the ſubtangent KH is every where of the ſame magnitude, the magnitude of the radius of curvature BG increaſes *ad infinitum* as the ordinate BK decreaſes.

293. It appears therefore that the radius of curvature of the logarithmic curve increaſes *ad infinitum* both when the ordinates to its axis increaſe *ad infinitum*, and when they decreaſe *ad infinitum*, and conſequently that its curvature decreaſes *ad infinitum* on both ſides of any given point in it, as was aſſerted in *Art.* 287.

Of the point of greateſt curvature in the logarithmic curve.

294. To determine the point at which the curvature of the logarithmic curve is greateſt, or the radius of curvature leaſt, we muſt determine the magnitude of the ordinate BK, or its proportion to the ſubtangent KH, when the proportion of BH^3 to $BK \times KH^q$ is a *minimum*, or is leſs than when BK has any other magnitude. For when the *ratio* of BH^3 to $BK \times KH^q$ is leſs than in any other caſe whatſoever, the *ratio* of BG to KH will be leſs than in any other caſe whatſoever; and conſequently, becauſe KH is every where of the ſame magnitude, and BG is always greater than KH, the magnitude of

BG will be leſs than in any other caſe whatſoever.

Now, if with H as a center and HK as a radius, we deſcribe the circular arch KC cutting BH in C, and from C draw CD, parallel to BK, meeting KH in D, 'tis evident BK will be the tangent, BH the ſecant, CD the ſine, and HD the coſine, of the arch BC, or of the angle BHK. Therefore HC or KH is to CD as BH to BK, and HC or KH is to HD as BH to KH, and conſequently $KH^q : HD^1 :: BH^q : KH^q$, and $KH^3 : CD \times HD^q :: BH^3 : BK \times KH^q$. Therefore the radius of curvature BG will be the leaſt poſſible when the *ratio* of KH^3 to $CD \times HD^q$ is the leaſt poſſible. But the *ratio* of KH^3 to $CD \times HD^q$ is the leaſt poſſible when $CD \times HD^q$ is at its greateſt magnitude, becauſe KH^3 is always of the ſame magnitude. Therefore the radius of curvature BG will be the leaſt poſſible when $CD \times HD$, (or $CD \times \overline{HC^q - CD^q}$ or $CD \times \overline{KH^q - CD}$) or $KH^q \times CD - CD^3$ is the greateſt poſſible. Now, if the angle BHK be conceived to increaſe gradually from *o* to a right angle, and conſequently its ſine CD to increaſe gradually from *o* till it becomes equal to the radius KH, 'tis evident the quantities $KH^q \times CD$ and CD^3 will both of them increaſe at the ſame time, from being at firſt infinitely ſmall till they each of them become equal to KH^3. Conſequently the difference of theſe quantities, or the quantity $KH^q \times CD - CD^3$, will, during the time of this increaſe, have increaſed from *o* to a certain finite magnitude, and have decreaſed from that magnitude to *o* again; for when CD is equal either to *o* or to KH, the quantity $KH^q \times CD - CD^3$ will evidently be $= o$. And this finite magnitude to which the quantity $KH^q \times CD - CD^3$ increaſes from *o*, and from which it again decreaſes to *o*, is the value of that quantity at the inſtant the infinitely ſmall increment of $KH^q \times CD$ becomes equal to the contemporary increment of CD^3. For as long as the former increment is greater than the latter, 'tis evident the quantity $KH^q \times CD - CD^3$ will increaſe continually; and when the latter increment becomes greater than the former, that quantity will continually decreaſe: conſequently at the inſtant the former increment ceaſes to be greater, and begins to be leſs, than the latter increment, that is, at the inſtant the two increments become equal, the quantity $KH^q \times CD - CD^3$ will ceaſe to increaſe and begin to decreaſe, or will have attained its greateſt magnitude. We muſt therefore ſuppoſe theſe increments to be equal, and the value of the ſine CD thence reſulting, will be that which makes the quantity $KH^q \times CD - CD^3$ the greateſt poſſible. Now, if KH be put $= r$, CD $= s$, and its infinitely ſmall increment $= \dot{s}$, we ſhall have $KH^q \times CD = rrs$, $CD^3 = s^3$, the infinitely ſmall increment of $KH^q \times CD = rrs$, and the contemporary increment of CD^3, or s^3, $= 3s^2\dot{s}$; therefore $rr\dot{s}$ is $= 3s^2\dot{s}$; whence rr is $= 3ss$, and $ss = \frac{rr}{3}$. Therefore the radius of curvature BG is the leaſt poſſible when ss is $= \frac{rr}{3}$, or CD^q

is $= \frac{HC^{q}}{3}$, or BK^{q} is $= \frac{BH^{q}}{3}$, and therefore when KH^{q} $(= BH^{q} - BK^{q})$ is $= BH^{q} - \frac{BH^{q}}{3} = \frac{2}{3}$ BH^{q} and conſequently BK^{q} is $= \frac{KH^{q}}{2}$; that is, the radius of curvature of the logarithmic curve is leaſt, and conſequently its curvature greateſt, in that point of it where the ſquare of the ordinate is equal to half the ſquare of the ſubtangent. QED.

295. It follows from hence, that the radius of curvature at the point of greateſt curvature in the logarithmic curve (which is about the point Ω in *fig.* 84.) is to the ſubtangent of the curve in the ſub-duplicate proportion of the numbers 27 and 4; or very nearly in the proportion of 2.598,557 to 1. For if we put $KH = a$, and $BK = x$, the general expreſſion of the radius of curvature BG, to wit, $\frac{BH^{3}}{BK \times KH}$, will be $\Big($ $=$

$$\frac{\overline{KH^{q} + BK^{q}} \times BH}{KH \times BK} =$$

$$\frac{\overline{KH^{q} + BK^{q}}}{KH \times BK} \times \sqrt{\overline{KH^{q} + BK^{q}}} =$$

$$\frac{aa + xx \times \sqrt{\overline{aa + xx}}}{ax} \text{ or } \frac{\overline{aa + xx}^{\frac{3}{2}}}{ax},$$ which,

when xx is $= \frac{aa}{2}$, becomes $= \frac{\sqrt{27}}{2} \times a$, or nearly $2.598,557 \times a$. For in this caſe we ſhall have $aa + xx = aa + \frac{aa}{2} = \frac{3aa}{2}$, $\overline{aa + xx}^{3}$

$= \frac{27a^6}{8}$, $\overline{aa+xx}^{\frac{1}{2}} = \frac{\sqrt{27}}{\sqrt{8}} \times a^3 = \frac{\sqrt{27}}{2\sqrt{2}} \times a^3$, $ax = a \times \frac{a}{\sqrt{2}} = \frac{aa}{\sqrt{2}}$, and $\frac{\overline{aa+xx}^{\frac{3}{2}}}{ax} = \frac{\sqrt{27}}{2\sqrt{2}} \times a^3 \times \frac{\sqrt{2}}{aa} = \frac{\sqrt{27}}{2} \times a =$ (nearly) $\frac{5.197114}{2} \times a = 2.598{;}557 \times a$.

296. The angle BHK, contained between the tangent BH of the logarithmic curve ABT and its axis SM, when the point B coincides with Ω, or the point of the greateſt curvature, is = 35°, 16′; and conſequently the angle HBK, contained between the tangent BH and the ordinate BK, is = 54°; 44′. For the radius is to the tangent of the angle BHK as KH to BK, that is, in the preſent caſe, as $\sqrt{2}$ to 1, or as 1 to $\frac{1}{\sqrt{2}} = \frac{1}{1.4142} = .7071 =$ tang. 35°, 16′.

297. When the angle BHK is = 45°, and conſequently the ordinate BK is equal to the ſubtangent KH, the radius of curvature BG is $= 2\sqrt{2} \times a$, or, nearly, $= 2.8284 \times a$. When the angle BHK is = 30°, and conſequently the ordinate BK (which is the tangent of that angle in a circle whoſe radius is KH, or a) is $= \frac{KH}{\sqrt{3}}$, or $\frac{a}{\sqrt{3}}$, the radius of curvature BG is $= \frac{8}{3} \times a = 2.6666$, &c. $\times a$. And when the angle BHK is = 60°, and conſequently BK is $= \sqrt{3} \times$ KH, or $\sqrt{3} \times a$, the radius of curvature BG is $= \frac{8}{\sqrt{3}}$

$\frac{8}{\sqrt{3}} \times a =$ (nearly) $4.6188 \times a$. All which values of BG are greater than $2.598,557 \times a$, or the value of BG when BKq is $= \frac{\text{KHq}}{2}$, and therefore ſerve to confirm the determination of the point of greateſt curvature given in *Art.* 295.

298. That the ſeveral values of BG are ſuch as are here aſſigned, will appear by computing the values of the expreſſion $\frac{\overline{aa + xx}\rceil^{\frac{3}{2}}}{ax}$ when x is equal to a; $\frac{a}{\sqrt{3}}$, and $\sqrt{3} \times a$, as follows. When x is $= a$, we have $aa + xx = 2aa$, $\overline{aa + xx}\rceil^{3} = 8\ a^{6}$, $\overline{aa + xx}\rceil^{\frac{3}{2}} = \sqrt{8} \times a^{3} = 2\sqrt{2} \times a^{3}$, and $a\,x = aa$; and conſequently $\frac{\overline{aa + xx}\rceil^{\frac{3}{2}}}{a} = \frac{2\sqrt{2} \times a^{3}}{aa}$ $= 2\sqrt{2} \times a =$ (nearly) $2 \times 1.4142 \times a =$ $2.8284 \times a$. When x is $= \frac{a}{\sqrt{3}}$, we ſhall have $xx = \frac{aa}{3}$, $aa + xx = \frac{4aa}{3}$, $\overline{aa + x}\rceil = \frac{64a^{6}}{27}$, $\overline{aa + xx}\rceil^{\frac{3}{2}}$ $\frac{8}{\sqrt{27}} \times a^{3}$, $= \frac{8}{3\sqrt{3}}$, $\times a^{3}$, and $ax = \frac{aa}{\sqrt{3}}$; and conſequently $\frac{\overline{aa + xx}\rceil^{3}}{ax} = \frac{8}{3\sqrt{3}} \times$ $a^{3} \times \frac{\sqrt{3}}{aa} = \frac{8}{3} \times a = 2.6666$, &c. $\times a$. And when x is $= \sqrt{3} \times a$, we ſhall have $xx = 3aa$, $aa + xx = 4aa$, $\overline{aa + xx}\rceil^{3} = 64a^{6}$, $\overline{aa + xx}\rceil^{\frac{3}{2}}$

= 8

$= 8a^3$, and $ax = \sqrt{3} \times aa$; and conſequently $\frac{\overline{aa+xx}^{\frac{3}{2}}}{ax} = \frac{8a}{\sqrt{3} \times aa} = \frac{8}{\sqrt{3}} \times a =$ (nearly) $\frac{8}{1.73205} \times a = 4.6188 \times a.$

Of the relation between two points of equal curvature in a logarithmic curve.

299. Since there is in every logarithmic curve a certain point at which its curvature is greater than at any other point of it, and on each ſide of which its curvature decreaſes, and the radius of curvature increaſes, *ad infinitum*, it follows that, if we take any point in this curve on one ſide of the point of greateſt curvature in it, there will always be another point in it on the other ſide of the point of greateſt curvature at which the radius of curvature will be of the ſame magnitude as at the former point. Thus, if in *Fig.* 84, we take the point B on the right-hand of the point Ω, or the point of greateſt curvature, there will be another point *b* in the curve ABT on the left-hand of the point Ω, at which the radius of curvature *bg* will be equal to the radius of curvature BG at the point B. Now, when one of theſe points (or the proportion of the ordinate that paſſes through it to the ſubtangent) is given, the other point, or the proportion of the ordinate that paſſes through it to the ſubtangent, may be determined in the following manner. Put the ſubtangent KH $= a$, the greater ordinate BK $= y$, and the leſſer ordinate

$bk = x$; and we ſhall have BG $= \frac{\overline{aa+yy}|^{\frac{3}{2}}}{ay}$, and $bg = \frac{\overline{aa+xx}|^{\frac{3}{2}}}{ax}$, and therefore becauſe bg is ſuppoſed to be $=$ BG, $\frac{\overline{aa+xx}|^{\frac{3}{2}}}{ax}$ will be $= \frac{\overline{aa+yy}|^{\frac{3}{2}}}{ay}$; therefore $\frac{\overline{aa+xx}|^{3}}{a^2x^2}$ will be $= \frac{\overline{aa+yy}|^{3}}{a^2y^2}$; or $\frac{a^6+3a^4x^2+3a^2x^4+x^6}{a^2x^2}$ will be $= \frac{a^6+3a^4y^2+3a^2y^4+y^6}{a^2y^2}$, and $a^6y^2+3a^4x^2y^2+3a^2x^4y^2+x^6y^2=a^6x^2+3a^4y^2x^2+3a^2y^4x^2+y^6x^2$, and $3a^2y^4x^2-3a^2x^4y^2+y^6x^2-x^6y^2=a^6y^2-a^6x^2=a^6 \times \overline{y^2-x^2}$, and conſequently $3a^2y^2x^2+y^4x^2+y^2x^4=a^6$. By the reſolution of this equation (which 'tis evident may be performed in the ſame manner as that of a quadratic equation) we may, when x is given, determine the magnitude of y, and, when y is given, we may determine the magnitude of x.

300. Thus, for inſtance, if x is $= \frac{1}{3}a$, we ſhall have $xx = \frac{aa}{9}$, $x^4 = \frac{a^4}{81}$, and $3a^2x^2 = 3a^2 \times \frac{a^2}{9} = \frac{a^4}{3}$, and conſequently a^6 $(= 3a^2y^2x^2+y^4x+y^2x^4) = \frac{a^4y^2}{3}+\frac{a^2y^4}{9}+\frac{a^4y^2}{81}=\frac{27a^4y^2}{81}+\frac{a^2y^4}{9}+\frac{a^4y^2}{81}=\frac{28a^4y^2}{81}+\frac{a^2y^4}{9}$. Therefore

8 a

$\frac{28a^2y^2}{9}+y^4$ is $=9a^4$, and $y^4+\frac{28a^2y^2}{9}+\frac{196a^4}{81}$ $=9a^4+\frac{196a^4}{81}=\frac{729a^4+196a^4}{81}=\frac{925a^4}{81}$. Conſequently $y^2+\frac{14a^2}{9}=\frac{30.4138}{9}\times a^2$, and $y^2=\frac{16.4138}{9}\times a^2$, and therefore y is $=$ $\frac{4.05}{3}\times a=1.35\times a$.

Again, let x be $=\frac{7}{10}\times a$, and we ſhall have xx $\frac{49}{100}\times aa$, $x^4=\frac{2401}{10{,}000}\times a^4$, and $3a^2x^2=$ $3a^2\times\frac{49a^2}{100}=\frac{147a^4}{100}$, and conſequently a^6 ($=$ $3a^2y^2x^2+y^4x^2+y^2x^4$) $=\frac{147a^4y^2}{100}+\frac{49a^2y^4}{100}+$ $\frac{2401a^4y^2}{10{,}000}=\frac{14700a^4y^2}{10{,}000}+\frac{49a^2y^4}{100}+\frac{2401a^4y^2}{10{,}000}$ $=\frac{17101a^4y}{10{,}000}+\frac{49a^2y^4}{100}$ Therefore $\frac{100a^4}{49}$ is $=y^4+\frac{17101a^2y^2}{4900}$, and y^4+ $\frac{17101a^2y^2}{4900}+\frac{292444101\times a^4}{4\times 24010000}=\frac{100a^4}{49}$ $+\frac{292444101\times a^4}{4\times 24010000}=2.0408\times a^4+3.0450\times a^4$ $=5.0858\times a^4$, and conſequently $y^2+\frac{17101}{9800}$ $\times a^2$ is $=2.2551\times a^2$, and $y^2=2.2551\times a^2$ -17101

$-\frac{17101 \times a^2}{9800} = 2.2551 \times a^2 - 1.7450 \times a^2$ $= .5101 \times a^2$, and $y = .714 \times a$.

Again, if x is $= \frac{a}{\sqrt{3}}$, we ſhall have $xx = \frac{aa}{3}$, $x^4 = \frac{a^4}{9}$, and $3a^2x^2 = 3a^2 \times \frac{a^2}{3} = a^4$, and conſequently a^6 ($= 3a^2y^2x^2 + y^4x^2 + y^2x^4$) $= a^4y^2 + \frac{a^2y^4}{3} + \frac{a^4y^2}{9} = \frac{9a^4y^2}{9} + \frac{a^2y^4}{3} + \frac{a^4y^2}{9} = \frac{10a^4y^2}{9} + \frac{a^2y^4}{3}$; therefore $3a^4$ is $= y^4 + \frac{10a^2y^2}{3}$, and $y^4 + \frac{10a^2y^2}{3} + \frac{25a^4}{9} = 3a^4 + \frac{25a^4}{9} = \frac{27a^4 + 25a^4}{9} = \frac{52a^4}{9}$, and conſequently $y^2 + \frac{5a^2}{3}$ is $= \frac{7.211,102}{3} \times a^2 = 2.403,700 \times a^2$, and y^2 is $= 2.403,700 \times a^2 - \frac{5a^2}{3} =$ $2\,403,700 \times a^2 - 1.666,666$, &c. $\times a^2 =$ $.737,034 \times a^2$, and conſequently $y = .858 \times a$,

In like manner, when y is given, we may find x. Thus, if y is $= d$, we ſhall have $y^2 = a^2$, $y^4 = a^4$, and a^6 ($= 3a^2y^2x^2 + y^4x^2 + y^2x^4$) $= 3a^4x^2 + a^4x^2 + a^2x^4 = 4a^4x^2 + a^2x$, and $x^4 + 4a^2x^2 = a^4$; therefore $x^4 + 4a^2x^2 + 4a^4$ is $= 5a^4$, and $x^2 + 2a^2 = \sqrt{5} \times a^2$ $= 2.23606 \times a^2$, whence x^2 is $= 2.23606 \times a^2 - 2a^2 = .23606 \times a^2$, and x is $= .485 \times a$.

If y is $= \sqrt{3} \times a$, we ſhall have $y^2 = 3aa$, $y^4 = 9a^4$, and $3a^2y^2 = 3a^2 \times 3a^2 = 9a^4$ and conſequently a^6 ($= 3a^2y^2x^2 + y^4x^2 + y^2x^4$) $= 9a^4x^2 + 9a^4x^2 + 3a^2x^4 = 18a^4x^2 + 3a^2$

$6a^2x^2 + 9a^4 = 9a^4 + \frac{a^4}{3} = \frac{27a^4 + a^4}{3} = \frac{28a^4}{3}$ $= 9.33333333 \times a^4$; therefore $x^2 + 3a^2$ is $= 3.0550\, a^2$, and $x^2 = 3.0550 \times a^2 - 3a^2 = .0550a^2$, and $x = .23 \times a$.

An obſervation concerning the decreaſe of curvature in a logarithmic curve, on the oppoſite ſides of the point of greateſt curvature.

301. From the foregoing equation, $3a^2x^2y^2 + y^4x^2 + y^2x^4 = a^6$, for determining the points of equal curvature on the oppoſite ſides of the point Ω, at which the curvature is greateſt, we may deduce the following obſervation concerning the decreaſe of the curvature of a logarithmic curve on the different ſides of the point of greateſt curvature; to wit, that its curvature decreaſes faſter, or the radius of curvature increaſes faſter, on that ſide of the point of greateſt curvature on which the ordinates to the axis increaſe than on the ſide on which the ordinates decreaſe; or, if we take any two points as *b* and Q in the logarithmic curve ABT on different ſides of the point Ω where the curvature is greateſt, and at equal diſtances, meaſured on the axis of the curve, from that point, ſo that the abſciſs EV intercepted between the ordinates ΩE and QV ſhall be equal to the abſciſs E*k* intercepted between the ordinates ΩE and *b k*, the radius of curvature at the point Q will be greater than the radius of curvature at the point *b*.

302. Now 'tis evident this will be true if the point B, at which the radius of curvature is of the ſame magnitude as at the point *b*, falls between the points Ω and Q. For if this be the

caſe, the radius of curvature at Q (the point Q being more diſtant than the point B from the point Ω where the curvature is greateſt) will be greater than the radius of curvature at B, and conſequently than the radius of curvature at *b*. We will therefore ſhew that the point B falls between the points Ω and Q, or that the abſciſs EK is leſs than the abſciſs EV, or than its equal the abſciſs E*k*, or (which amounts to the ſame thing, becauſe the abſciſſes EK, E*k* are meaſures of the *ratios* of BK to OE and OE to *bk*) that the *ratio* of BK to OE is leſs than the *ratio* of OE to *bk*, or that the *ratio* of BK^q to OE^q, or $\frac{KH^q}{2}$, is leſs than the *ratio* of OE^q, or $\frac{KH^q}{2}$, to bk^q, or that BK^q is leſs than a third proportional to bk^q and $\frac{KH^q}{2}$, or than $\frac{KH^4}{4bk^q}$ or that yy is leſs than $\frac{a^4}{4xx}$.

303. In order to demonſtrate this aſſertion, we muſt reſolve the equation $3a^2y^2x^2 + y^4x^2 + y^2x^4 = a^6$ in general terms, or find a general expreſſion of the value of yy in the following manner. Let the proportion of a to x be repreſented in general by that of the numbers m and n, ſo that x ſhall be $= \frac{n}{m} \times a$. Then we ſhall have $x^2 = \frac{n^2a^2}{m^2}$, $x^4 = \frac{n^4a^4}{m^4}$, $3a^2y^2x^2 = 3a^2y^2 \times \frac{n^2a^2}{m^2} = \frac{3n^2a^4y^2}{m^2}$, $y^4x^2 = y^4 \times \frac{n^2a^2}{m^2} =$

$\frac{n^2a^2y^4}{m^2}$, and $y^2x^4 = y^2 \times \frac{n^4a^4}{m^4} = \frac{n^4a^4y^2}{m^4}$; therefore a^6 $(=3a^2y^2x^2+y^4x^2+y^2x^4)$ is $= \frac{3n^2a^4y^2}{m^2} + \frac{n^2a^2y^4}{m^2} + \frac{n^4a^4y^2}{m^4} = \frac{3m^2n^2a^4y^2+n^4a^4y^2}{m^4} + \frac{n^2a^2y^4}{m^2}$, and $a^4 = \frac{3m^2n^2a^2+n^4a^2}{m^4} \times y^2 + \frac{n^2y^4}{m^2}$, and $\frac{m^2a^4}{n^2} = \frac{3m^2a^2+n^2a^2}{m^2} \times y^2 + y^4$. Therefore $y^4 + \frac{3m^2a^2+n^2a^2}{m^2} + \frac{9m^4a^4+6m^2n^2a^4+n^4a^4}{4m^4}$ is $=$

$$\frac{m^2a^4}{n^2} + \frac{9m^4a^4+6m^2n^2a^4+n^4a^4}{4m^4} =$$

$\frac{4m^6a^4+9m^4n^2a^4+6m^2n^4a^4+n^6a^4}{4m^4n^2}$, and consequently $y^2 + \frac{3m^2a^2+n^2a^2}{2m^2} =$

$\frac{\sqrt{4m^6a^4+9m^4n^2a^4+6m^2n^4a^4+n^6a^4}}{2m^2n}$, and y^2

$$= \frac{\sqrt{4m^6a^4+9m^4n^2a^4+6m^2n^4a^4+n^6a^4}}{2m^2n}$$

$- \frac{3m^2a^2-n^2a^2}{2m^2}$. Q.E.D.

304. As an inſtance of this method of finding the value of y, we will reſume the firſt of the examples reſolved above in *Art.* 300, in which x was taken $= \frac{1}{3} \times a$. Here therefore we ſhall have $m = 3$, $n = 1$, $n^2 = 1$, $n^6 = 1$, $m^2 = 9$, $m^4 = 81$, and $m^6 = 729$, and conſequently

$4m^6a^4 = 4\times729\times a^4 = 2916a^4$, $9m^4n^2a^4 = 9\times81\times1\times a^4 = 729a^4$, $6m^2n^4a^4 = 6\times9\times1\times a^4 = 54a^4$, $n^6a^4 = a^4$, and therefore $\sqrt{4m^6a^4+9m^4n^2a^4+6m^2n^4a^4+n^6a^4} = \sqrt{2916+729+54+1}\times a^4 = \sqrt{3700a^4} = 60.8276\times a^2$; which divided by $2m^2n$, or $(2\times9\times1=)$ 18 gives $3.3793\times a^2$; from which if we take $\frac{3m^2a^2+n^2a^2}{2m^2}$, or $\left(\frac{3\times9\times a^2+a^2}{2\times9} = \frac{27a^2+a^2}{18} = \frac{28a^2}{18} =\right)$ $\frac{14a^2}{9}$, or $1.5555\times a^2$, we ſhall have $y^2 = 1.8238\times a^2$, and conſequently y will be $= 1.35\times a$; which, agreeing exactly with the value of y obtained in this example in *Art.* 300, by reſolving the equation $3a^2y^2x^2+y^4x^2+y^2x^4 = a^6$ in particular numbers, ſerves as a confirmation of the foregoing general expreſſion of the value of yy, or as a proof that it is rightly computed.

305. Now becauſe x is $= \frac{na}{m}$, and conſequently $xx = \frac{n^2a^2}{m^2}$, 'tis evident $\frac{a^4}{4xx}$ will be $= \frac{a^4}{4}\times\frac{m^2}{n^2a^2} = \frac{m^2a^2}{4n^2}$. We are therefore to demonſtrate, that the foregoing expreſſion of the value of yy is leſs than $\frac{m^2a^2}{4n^2}$; which may be thus ſhewn.

306. Since when the point b coincides with the point n, the value of bk^q, or xx, is only $= \frac{aa}{2}$, it follows, that in the preſent caſe, in

which the point b is ſuppoſed not to coincide with Ω, xx muſt be leſs than $\frac{aa}{2}$; conſequently nn will be leſs than $\frac{mm}{2}$, and $2nn$ than mm. Therefore the ſquare of $mm - 2nn$ muſt be an affirmative quantity, or its affirmative parts m^4 and $4n^4$ muſt be greater than its negative part $4m^2 n^2$; therefore $\frac{m^4}{4} + n^4$ will be greater than m^2n^2, and $\overline{\frac{m^4}{4} + n^4} \times m^4 a^4$ than $m^2n^2 \times m^4 a^4$, or $\frac{m^8 a^4}{4} + m^4n^4a^4$ than $m^6n^2a^4$ Therefore adding $3m^6n^2a^4 + 9m^4n^4a^4 + 6m^2n^6a^4 + n^8a^4$, to both ſides, it follows that $\frac{m^8a^4}{4} + 3m^6n^2a^4 + 10m^4n^4a^4 + 6m^2n^6a^4 + n^8a^4$, will be greater than $4m^6n^2a^4 + 9m^4n^4a^4 + 6m^2n^6a^4 + n^8a^4$. Conſequently

$$\frac{\frac{m^8a^4}{4} + 3m^6n^2a^4 + 10m^4n^4a^4 + 6m^2n^6a^4 + n^8a^4}{4m^4n^4}$$

is greater than

$$\frac{4m^6n^2a^4 + 9m^4n^4a^4 + 6m^2n^6a^4 + n^8a^4}{4m^4n^4} \text{ or}$$

than $$\frac{4m^6a^4 + 9m^4n^2a^4 + 6m^2n^4a^4 + n^6a^4}{4m^4n^2};$$

and conſequently the ſquare-root of the former of theſe quantities is greater than the ſquare-root of the latter, that is, $\dfrac{\dfrac{m^4a^2}{2}+3m^2n^2a^2+n^4a^2}{2m^2n^2}$, or $\dfrac{m^2a^2}{4n^2}+\dfrac{3m^2a^2+n^2a^2}{2m^2}$, is greater than $\dfrac{\sqrt{4m^6a^4+9m^4n^2a^4+6m^2n^4a^4+n^6a^4}}{2m^2n}$.

Conſequently $\dfrac{m^2a^2}{4n^2}$ is greater than $\dfrac{\sqrt{4m^6a^4+9m^4n^2a^4+6m^2n^4a^4+n^6a^4}}{2m^2n}$ $\dfrac{-3m^2a^2-n^2a^2}{2m^2}$, which is the expreſſion found above for the value of yy; and therefore $\dfrac{m^2a^2}{4n^2}$, or $\dfrac{a^4}{4xx}$ is greater than yy. QED.

The point of greateſt curvature determined by another method different from that in Art. 294.

307. There is one thing further, which it may be worth our while to obſerve concerning the equation $3a^2y^2x^2+y^4x^2+y^2x^4=a^4$ found above for determining the proportion of the ordinates BK, *bk*, at the points B, and *b*, that have the ſame degree of curvature, to wit, that it affords a method of finding the point Ω of greateſt curvature different from that uſed in *Art.* 294. For we need only ſuppoſe the points B and *b* to approach continually to each other, till they coincide in the point Ω, and the ordinates BK and

bk, or y and x, become equal to each other, and to the ordinate ΩE, and the value of x in the equation $3a^2y^2x^2 + y^4x^2 + y^2x^4 = a^6$, which in this caſe will be $3a^2x^4 + x^6 + x^6 = a^6$, or $x^6 + \frac{3a^2x^4}{2} = \frac{a^6}{2}$ will be the magnitude of the ordinate ΩE that paſſes through the point Ω of greateſt curvature. Now the value of xx in this equation is $\frac{aa}{2}$, as will be evident by ſubſtituting $\frac{aa}{2}$ in its ſtead in the terms x^6 and $\frac{3a^2x^4}{2}$; for if this be done, we ſhall have $x^6 = \frac{a^6}{8}$, and $\frac{3a^2x^4}{2} = \frac{3a^2}{2} \times \frac{a^4}{4} = \frac{3a^6}{8}$, and $x^6 + \frac{3a^2x^4}{2} = \frac{a^6}{8} + \frac{3a^6}{8} = \frac{4a^6}{8} = \frac{a^6}{2}$, as in the equation whoſe root was to be found. It appears therefore, that the determination of the point Ω of greateſt curvature, derived from this equation, agrees exactly with the determination of the ſame point, obtained by another method in *Art.* 294, and thereby ſerves as a confirmation of that determination, or as a farther proof that the point Ω is rightly aſſigned.

Here we put an end to this 10th *Corollary*.

Corol. 11. We now proceed by the help of the foregoing *Corollaries* to determine the areas of the figures of ſines, verſed ſines, and chords. And firſt we will conſider the figure of ſines.

Let ABD (*fig.* 85) be the arch of a ſemi-circle whoſe center is C; AB the arch of a quadrant; AE, AG, AN, any arches leſs than AB; and AF, AH, any arches greater than AB, but leſs than the ſemi-circle ABD; and EM, Gt, NS, $F\gamma$, $H\delta$, the ſines of theſe arches. Draw the right line $a\,d$ (*Fig.* 86) equal to the ſemicircular arch ABD, and take the ſeveral portions of this line, ae, ag, an, ab, af, ah, reſpectively equal to the arches AE, AG, AN, AB, AF, AH; and through the points e, g, n, b, f, h, draw the lines eT, gV, np, bK, $f\gamma$, $h\delta$, at right angles to ad, and reſpectively equal to the ſines EM, Gt, NS, BC, $F\gamma$, $H\delta$, of the arches AE, AG, AN, AB, AF, AH; and let the curve $aTUKd$ be drawn, in which the extremities T, V, p, K, γ, δ, of theſe perpendiculars are ſituated. The figure $aTUKda$ is called the figure of ſines.

309. Now concerning this figure, we may obſerve in the firſt place that the latter half of it, to wit, $K\gamma\delta db$, is exactly equal and ſimilar to the former half of it, $KUTab$. For, as the ſines of arcs that differ equally from a quadrant, or whereof the one exceeds a quadrant as much as the other falls ſhort of it, are equal, 'tis evident that the ordinates of the figure of ſines $aTUKda$ at equal diſtances from the middle ordinate bK will be equal to each other: thus, for inſtance, if bf is $= bn$, the ordinate $f\gamma$ will be equal to the ordinate $n\,p$. Conſequently, if the latter half $K\gamma\delta db$ of the figure $aTUKda$ was laid upon the former half $KUTab$ of that figure, it would exactly cover, or coincide

with it, and therefore is exactly equal and ſimilar to it.

2*dly*, The curve *a*TUK*γδd* of the figure of ſines *a*TUK*da* is every where concave towards its baſe *a d*, It appears from the demonſtrations in *Art.* 198, and 280, in which it was ſhewn that the logarithmic curve, and the curves of the figures of tangents and ſecants were convex towards their baſes, that any curve in general is convex towards its baſe, when, if two contiguous portions of the baſe be taken equal to each other, and the correſpondent ordinates be drawn, the difference of the greateſt and middle ordinates is greater than the difference of the middle and leaſt ordinates: and by the ſame kind of reaſoning it will follow, that, if the difference of the greateſt and middle ordinates is leſs than the difference of the middle and leaſt ordinates, the curve will be concave toward its baſe. Now, if the portions *eg*, *gn* of the ſemi-baſe *ab* of the figure of ſines *a*TUK*da* (*Fig.* 86.) be taken equal to each other; the correſpondent arches EG, GN will be equal to each other, and therefore (by *Prop.* 24, *Corol.* 1.) the difference of the ſines NS, G*t* will be leſs than the difference of the ſines G*t*, EM; therefore the difference of the ordinates *np*, *g*V, which are reſpectively equal to the ſines NS, G*t*, will be leſs than the difference of the ordinates *g*V, *e*T, which are reſpectively equal to the ſines G*t*, EM. Therefore the curve *a*TUK is concave towards *ab*; and conſequently the curve K*δd*, which is exactly equal and ſimilar to *a*TVK, is concave to-

wards

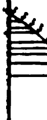


Fig. 82.

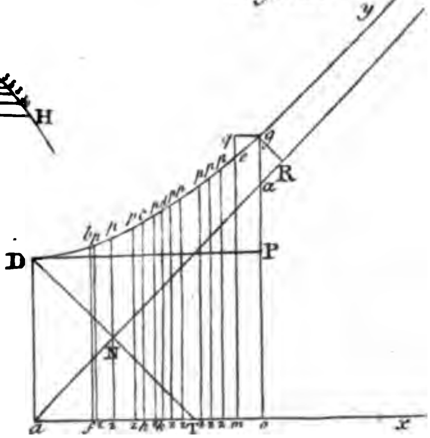
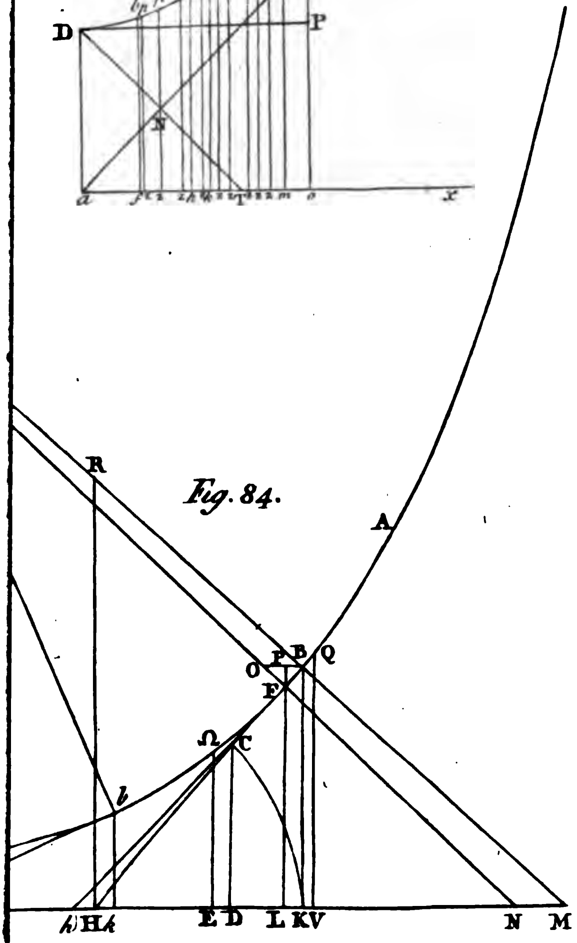


Fig. 84.



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wards *bd*; therefore the whole curve *a*TUK*bd* is concave towards the baſe *a d*.QED.

3*dly*, The area of any portion, as *a*TV*pna*, of the figure of ſines *a*TUK*da* is equal to the rectangle under the radius of the circle ABD and the verſed ſine AS of the arch AN to which the baſe *a n* of that portion is equal; or, if A *s* (*Fig.* 87.) is taken equal to the verſed ſine AS, and A*λ* be drawn perpendicular to A*s* and equal the radius CA, and the rectangle A*λps* be compleated, the area *a*TV*pna* will be equal to the rectangle A*λps*.

In the circular arch AN take the point L infinitely near to the point N, and draw the ſine LR; and in the line *an* take *ln* equal to the arch LN, and draw the ordinate *lk*; and from *k* draw *kq*, parallel to *a n*, meeting *np* in *q*. *Laſtly*, in the line A *s* (*Fig.* 87.) take *rs* = RS, the difference of the verſed ſines AR, AS, and draw *re*, parallel to A*λ*; meeting *λp* in *e*. Then, ſince by *Corol.* 6. the infinitely ſmall difference LN of of the arches AL, AN, is to RS, the difference of the verſed ſines AR, AS, of thoſe arches, as the radius AC to the ſine LR, and (by the nature of the figure of ſines) the ordinate *l k* is equal to the ſine LR, and the lines *re*, *rs*, *ln*, are reſpectively equal to the radius AC, the difference RS of the verſed ſines AR, AS, and the difference LN of the arches AL, AN, it follows that *ln* : *rs* :: *re* : *lk*; therefore the rectangle *r e p s* is equal to the rectangle *l k q n*. But, becauſe the ordinates *l k*, *n p* are infinitely near to each other, the mixtilinear area *l k p n* is equal to the rectangle *l k q n*; therefore the mixtilinear area *l k p n* is equal to the rectangle *r e p s*. And, in the ſame manner it may be ſhewn that, if the area *a*TV*pna* be divided into an infinite number of ſuch little mixtilinear areas as *l k p n*, and the rectangle A *λ p s* be divided into the ſame number of correſpondent rectangles, ſuch as the rectangle *r e p s*, each of the rectangles into which the rectangle A*λps* is divided, will be equal to the correſponding mixtilinear area, or portion, of the area *a*TV*pna*. Therefore the whole area *a*TV*pna* is equal to the whole rectangle A*λps*. QED.

This demonſtration, 'tis eaſy to perceive, is equally juſt, whether the arch AN be leſs or greater than a quadrant, although the figures repreſent the former caſe only. For it is evident from *Corol.* 6, that in both caſes the proportion of LN to RS, or of *ln* to *rs*, is equal to that of AC to LR, or of *r e* to *l k*; which is the foundation of the foregoing demonſtration.

310. It follows from this demonſtration, that the area of half the figure of ſines *a*TVK*da*, or of the portion *a*TUK*ba*, is equal to the ſquare of the radius, and the area of the whole figure of ſines *a*TUK*da* is equal to the rectangle under the radius and diameter.

311. It follows alſo that the area *p*K*bn* intercepted between the portion *bn* of the baſe, the arch *p*K, the ordinate *np*, and the middle, or greateſt, ordinate *b*K, is equal to the rectangle under the radius CA and the difference of that radius from the verſed ſine AS, or to the rectangle under the radius CA and coſine CS of the arch AN, or ſine of the arch BN to which the baſe *bn* of the portion *p*K*bn* is equal.

312. *Note*, Dr. *Keill* has made uſe of this figure in computing the heat of the ſun in a particular caſe mentioned in his examination of *Burnet*'s *theory of the earth*, chap. vii. edit. 2d, p. 122. And we may further obſerve concerning this figure, that, if the ordinates *np*, *b*K, *fy*, &c. be all diminiſhed in the ſame proportion, the curve *a*TVK$\gamma\delta d$ will be that curve to which Dr. *Smith* and other mathematicians have given the name of the *harmonical curve*, and in one of which they have demonſtrated, that a muſical ſtring actually vibrates. [See *Smith*'s *Harmonics*, ſect. 11.]

313. *Corol.* 12. In *Fig.* 88, let the right line *a d* be taken equal to the ſemi-circular arc ABD in *Fig.* 85, the line *a b* to the quadrant AB, and the lines *ae*, *ag*, *an*, *af*, *ah*, to the arches AE, AG, AN, AF, AH: and through the points *e*, *g*, *n*, *b*, *f*, *h*, *d*, draw the ordinates *e*P, *g*Q, *np*, *b*K, *f*F, *h*H, and *d*T at right angles to the line *a d*, and reſpectively equal to the verſed ſines AM, A*t*, AS, AC, Aγ, Aδ, AD, of the arches AE, AG, AN, AB, AF, AH, AD; and let the curve *a*PQ*p*KFHT be drawn, in which the extremities of all theſe ordinates are ſituated: the figure *ap*KT*da* is called the figure of verſed ſines.

314. Now, concerning this figure it muſt be obſerved that the curve *ap*KT is not in every part of it bent the ſame way; but the firſt part of it, *ap*K, correſponding to the ſemi-baſe *a b*, or to the quadrant A B, is convex towards the line *a d*, and the ſecond part of it, KFHT, correſponding to the latter ſemi-baſe *bd*, or to the quadrant BD, is concave towards the line *a d*.

For ſince, by *Corol.* 6, the infinitely ſmall increment of a circular arc is to the contemporary increment of its verſed ſine, as the radius of the circle to the ſine of that arc, it follows that, if in the ſemi-circular arch ABD in *Fig.* 85, we take three equi-different arches, whoſe differences are infinitely ſmall in compariſon of the arches themſelves, the difference of the verſed ſines of the greateſt and middle arches will be to the difference of the verſed ſines of the middle and leaſt arches as the ſine of the middle arc is to the ſine of the leaſt arc: but the ſine of the middle arc is greater than the ſine of the leaſt arc as long as theſe arcs are leſs than a quadrant; and afterwards, when theſe arcs are greater than a quadrant, it is greater than the ſine of the leaſt arc: therefore, when theſe arcs are leſs than a quadrant, the difference of the verſed ſines of the greateſt and middle arcs is greater than the difference of the verſed ſines of the middle and leaſt arcs; and when theſe arcs are greater than a quadrant, the difference of the verſed ſines of the greateſt and middle arcs is leſs than the difference of the verſed ſines of the middle and leaſt arcs. Conſequently, if we take three equi-different portions of the line *a d* (*Fig.* 88.) whoſe differences are infinitely ſmall in compariſon of the lines themſelves, and through the extremities of theſe lines draw the correſpondent ordinates, the difference of the greateſt and middle ordinates will be greater than the difference of the middle and leaſt ordinates as long as theſe abſciſſes of the baſe *ad* are leſs than *a b*; and when theſe abſciſſes are greater than *a b*, the difference of the greateſt and middle ordinates will be leſs than the difference of the middle and leaſt ordinates. Therefore the portion *ap*K of the curve *ap*KFT, which correſponds to the ſemi-baſe *a b*, is convex towards *a d*; and the portion KFT of that curve, which correſponds to the ſemi-baſe *b d*, is concave towards *a d*. QED.

315. The area of this figure is determined in the following manner. If the abſciſs *a n* of the baſe *a d* be leſs than the ſemi-baſe *a b*, the area of the portion *a*PQ*pna*, correſponding to that abſciſs, will be equal to the exceſs of the rectangle under the radius CA and abſciſs *a n*, or arch A N, above the rectangle under the radius CA and NS the ſine of the arch AN; and conſequently the area of the portion *a*PQ*p*K*ba*, correſponding to the ſemi-baſe *a b*, is equal to the exceſs of the rectangle under the radius CA and the ſemi-baſe *a b*, or quadrantal arch A B, above the ſquare of the radius C A. If the abſciſs *a b* of the baſe *a d* is greater than the ſemi-baſe *a b*, the area of the portion *ap*K*ba*, correſponding to that abſciſs, is equal to the ſum that ariſes by adding to the area of the portion *ap*K*ba*, that correſponds to the ſemi-baſe *a b*, (or to the exceſs of the rectangle under the radius AC and the quadrantal arch AB above the ſquare of the radius CA) the ſum of the rectangle under the radius CA and portion *b b* of the baſe *a d*, or arch B H, and the rectangle under the radius CA and the exceſs of the radius CA above H*δ* the ſine of the arch ABH; or, in other words, the area of the portion KH*bb*, correſponding to the portion *b b* of the baſe *a d*, is equal to the ſum of the rectangle under the radius CA and the arch BH and the rectangle under the radius CA

and the exceſs of the radius CA above H*d* the ſine of the arch ABH.

In order to demonſtrate theſe determinations, take *a b*, in *Fig*. 89, equal to *a b* the ſemi-baſe of *Fig*. 88, or to the quadrantal arch AB, and draw *a*K perpendicular to *a b*, and equal to the radius AC; and compleat the rectangle *a*KC*b*. Alſo take *rb*, in *Fig*. 90, equal to the radius AC, and compleat the ſquare *r*KB*b*. In the arch AN take the point L infinitely near to the point N, and draw the ſine LR, and the line LP, parallel to AS, meeting NS in P; and take *ln*, in *Fig*. 88, equal to the infinitely ſmall arch LN, and draw the ordinate *l k* meeting the curve *a*PQ*p* in *k*; and from *k* draw *kq*, parallel to *a b*, meeting *n p* in *q*. Laſtly, take *a n*, and *a l*, in *Fig*. 89, reſpectively equal to *a n* and *a l* in *Fig*. 88, or to the arches A N, A L, and thro' the points *n* and *l* draw the lines *n* N, *l* L, parallel to *a* K, meeting KC in N and L; and in *Fig*. 90, take *r p* and *r l* reſpectively equal to the ſines NS, LR, of the arches AN, AL, and through the points *r*, *p*, draw the lines *p*N, *l*L, parallel to *r*K, meeting KB in N and L.

Then 'tis evident in the firſt place, that *l k*, in *Fig*. 88, will be equal to the verſed ſine AR; *l n* in *Fig*. 89, will be = LN; and *l p* in *Fig*. 90, will be = PN. Further, becauſe by *Corol*. 6, we have RS : LN :: LR : CA, and by *Corol*. 2. LN : PN :: CA : RC, it follows, *ex æquo*, that RS : PN :: LR : RC; conſequently LN is $= \frac{CA \times RS}{LR}$, and PN $= \frac{RC \times RS}{LR}$. Therefore *ln* x *lk* will be = LN x AR =

CA x

$\frac{CA \times RS}{LR} \times AR = \frac{CA \times RS}{LR} \times \overline{CA - RC}$ $= CA \times \frac{CA.RS}{LR} - CA \times \frac{RC.RS}{LR} = CA \times LN - CA \times PN = lL \times ln - lL \times lp$. But, becauſe the ordinates *np*, *lk* are infinitely near each other, the mixtilinear area *lkpn* will be equal to the rectangle under *lk*, and *ln*; therefore the mixtilinear area *lkpn* is $= lL \times ln - lL \times lp$, and conſequently $lL \times ln$ is $= lkpn + lL \times lp$. And in the ſame manner it may be ſhewn that, if the area *a*P*pna* is divided into an infinite number of infinitely ſmall mixtilinear areas, ſuch as *lkpn*, and the rectangles *a*KN*n* and *r*KN*p* are divided into the ſame number of correſponding little rectangles, ſuch as *l*LN*n* and *l*LN*p*, each of the little rectangles that compoſe the rectangle *a*KN*n* will be equal to the ſum of the correſponding portions of the area *a*P*pna* and rectangle *r*KN*p*: conſequently the whole rectangle *a*KN*n* will be equal to the ſum of the area *a*P*pna* and the rectangle *r*KN*p*, and therefore the area *a*P*pna* will be equal to the exceſs of the rectangle *a*KN*n* above the rectangle *r*KN*p*; which was the firſt thing to be demonſtrated.

316. Hence it follows, that the area *a*PK*ba*, correſponding to the ſemi-baſe *ab*, is equal to the exceſs of the rectangle *a*KC*b* above the rectangle *r*KB*b*, or to the exceſs of the rectangle under the quadrantal arch AB and radius CA above the ſquare of the radius CA. For when *an* becomes equal to *ab*, *rp* becomes equal to *rb*.

317. The magnitude of the area KH*bb*, correſponding to the portion *b b* of the baſe *a d*, being equal to the difference of the areas HT*db*, and KT*db*, will be determined when thoſe areas are determined. Now, theſe areas may be found as follows.

Take *b d* in *Fig.* 91, equal to *bd* in *Fig.* 88, or to the quadrantal arch BD, and draw *b*C perpendicular to *bd*, and equal to the radius CA, and compleat the rectangle *b*CΔ*d*. Alſo take *b*Π in *Fig.* 92, equal to the radius CA, and compleat the ſquare *b*BΛΠ. In the arch HD take the point φ infinitely near to the point H, and draw the ſine φσ, and the line φπ, parallel to Aσ, meeting Hδ in π; and take *h*φ in *Fig.* 88, equal to the infinitely ſmall arch Hφ, and draw the ordinate φθ meeting the curve HT in θ; and from θ draw θλ, parallel to *bd*, meeting *b*H (produced) in λ. Laſtly, take *db* and *d*φ in *Fig.* 91, reſpectively equal to *db* and *d*φ in *Fig.* 88, or to the arches DH and Dφ, and through the points *b*, φ, draw the lines *b o*, φω, parallel to *b*C, meeting CΔ in *o* and ω; and in *Fig.* 92, take Πι, Πζ, reſpectively equal to the ſines Hδ, φσ of the arches DH, Dφ, and through the points ι, ζ, draw the lines ιη, ζ*z*, parallel to *b*B, meeting BΛ in η and *z*.

Then 'tis evident in the firſt place, that φθ in *Fig.* 88, will be = to the verſed ſine Aσ, φ*b* in *Fig.* 91, will be = to the arch φH, and ζι in *Fig.* 92. will be = Hπ. Further, by *Corol.* 6, we ſhall have σδ : φH :: φσ : CA, and by *Corol.* 2, φH : πH :: CA : Cσ, and conſequently *ex equo* σδ : πH :: φσ : Cσ; therefore φH is =

is $= \frac{CA \times \sigma\delta}{\phi\sigma}$, and $\pi H = \frac{C\sigma \times \sigma\delta}{\phi\sigma}$. Therefore $b\phi \times \phi\theta$ will be $= \phi H \times A\sigma = \frac{CA \times \sigma\delta}{\phi\sigma} \times A\sigma = \frac{CA \times \sigma\delta}{\phi\sigma} \times \overline{CA + C\sigma} = CA \times \frac{CA . \sigma\delta}{\phi\sigma} + CA \times \frac{C\sigma . \sigma\delta}{\phi\sigma} = CA \times \phi H + CA \times \pi H = \phi\omega \times \phi b + \zeta z \times \zeta\varepsilon$. But, becauſe the ordinates bH, $\phi\theta$ are infinitely near each other, the mixtilinear area $Hb\phi\theta$ will be equal to the rectangle under $b\phi$ and $\phi\theta$; therefore the mixtilinear area $Hb\phi\theta$ is equal to $\phi\omega \times \phi b + \zeta z \times \zeta\varepsilon$. And in the ſame manner it may be ſhewn that, if the area $HTdb$ is divided into an infinite number of infinitely ſmall mixtilinear areas, ſuch as $H\theta\phi b$, and the rectangles $bo\Delta d$ and $\varepsilon\eta\Lambda\Pi$ are divided into the ſame number of correſponding little rectangles, ſuch as $\phi\omega o b$ and $\zeta z\eta\varepsilon$, each of the portions into which the area $HTdb$ is divided will be equal to the ſum of the correſponding portions of the rectangles $bo\Delta d$ and $\varepsilon\eta\Lambda\Pi$. Conſequently the whole area $HTdb$ will be equal to the ſum of the two rectangles $bo\Delta d$ and $\varepsilon\eta\Lambda\Pi$, or to $CA \times DH + CA \times H\delta$. QEI.

318. Hence it follows, that the area of the portion $KHTdb$ of the figure of verſed ſines, correſponding to the latter ſemi-baſe bd, is equal to $bC\Delta d + bB\Lambda\Pi$ or $CA \times BD + CAq$, that is, to the ſum of the rectangle under the radius and the arch of a quadrant and the ſquare of the radius. For when the point b in *Fig*. 88, coincides with the point b, the point b in *Fig*. 91 will coincide with the point b, and the point ε in *Fig*. 92 will coincide with the point b.

319. From theſe two determinations it follows, that the area KH*bb* is ($= KTdb - HTdb = bC\Delta d + bb\Lambda\Pi - bp\Delta d - \eta\epsilon\Lambda\Pi$) $= bCob + bB\eta\epsilon = bC \times bb + bB \times \overline{b\Pi - \epsilon\Pi} = CA \times BH + CA \times \overline{CA - H\delta}$; which was the ſecond thing to be demonſtrated.

320. 'Tis evident from theſe concluſions, that the whole figure of verſed ſines *ap*KHT*da* is ($= apKba + KHTdb = CA \times$ arc $AB - CAq +$ $CA \times$ arc $BD + CAq = CA \times$ arc $AB + CA \times$ arc BD) equal to CA × arc ABD, or to the rectangle under the radius CA and the arch ABD of the ſemi-circle ABDA. And 'tis further evident, that the portion KHT*db* of this figure, correſponding to the latter ſemi-baſe *bd*, exceeds the portion *ap*K*ba*, correſponding to the former ſemi-baſe *a b*, by twice the ſquare of the radius CA.

321. *Corol.* 13. Let ABD (*Fig.* 93.) be the arch of a ſemi-circle whoſe center is C, AB the arch of a quadrant, AE, AG, AN, any arches leſs than a quadrant, and AF, AH, any arches greater than a quadrant, but leſs than the ſemicircle ABD. And draw the diameter AD, and the chords of the arches AE, AG, AN, AB, AF, AH. This done, take the right line *ad* (*Fig.* 94.) equal to the ſemicircular arch ABD, and in it take the lines *ae*, *ag*, *an*, *ab*, *af*, *ah*, reſpectively equal to the arches AE, AG, AN, AB, AF, AH; and at the points *e*, *g*, *n*, *b*, *f*, *h*, draw the right lines *e*P, *g*Q, *np*, *b*K, *f*F, and *h*H, at right angles to the line *ad*, and reſpectively equal to the chords of the arches AE, AG, AN, AB, AF, AH, and *d*Γ equal to the diameter AD; and draw the curve *a*PQ*p*KFHT, in

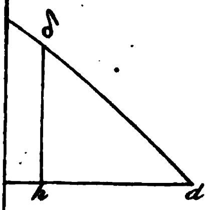


Fig. 90.

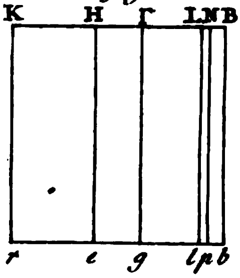


Fig. 89.

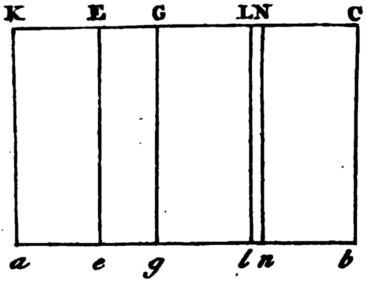
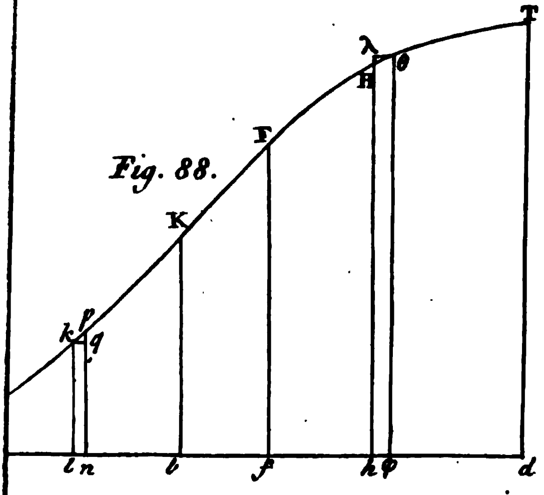


Fig. 88.



I. Mynde fc.

in which the extremities of all theſe perpendiculars are ſituated. The figure *ap*KFT*da* is called the figure of chords.

322. The curve *ap*KHT is every where concave towards the baſe *a d*. For ſince, by *Corol.* 7, the infinitely ſmall increment of a circular arc is to the contemporary increment of its chord as the diameter of the circle to the chord of the ſupplement of that arc to a ſemi-circle, it follows that, if in the ſemi-circular arch ABD we take three equi-different arches, whoſe differences are infinitely ſmall in compariſon of the arches themſelves, the difference of the chords of the greateſt and middle arches will be to the difference of the chords of the middle and leaſt arches as the chord of the ſupplement of the middle arc to the chord of the ſupplement of the leaſt arc to a ſemi-circle, and therefore will be leſs than the difference of the chords of the middle and leaſt arches. Conſequently, if we take three equi-different portions of the line *a d* (*Fig.* 94.) whoſe differences are infinitely ſmall in compariſon of the lines themſelves, and thro' the extremities of theſe lines draw the correſpondent ordinates, the difference of the greateſt and middle ordinates will be leſs than the difference of the middle and leaſt ordinates. Therefore the curve *ap*KT will be concave towards the baſe *a d*. QED.

323. The area of any portion *apna* of this figure is equal to the rectangle under the diameter AD, and the difference of that diameter and the chord of DN the ſupplement of the arch AN to a ſemi-circle.

In the arch AN (*Fig.* 93.) take the point L infinitely near to the point N, and draw the chords AL, DL, DN, and let y be the point in which the chord DL interſects the chord AN. In *Fig.* 94, take $l\,n$ equal to the arch L N, and draw the ordinate $l\,k$; and from k draw $k\,r$, parallel to $a\,n$, meeting $n\,p$ in r. And laſtly, take $\alpha\delta$, in *Fig.* 95, equal to the diameter AD, and compleat the ſquare $\alpha \mathrm{MB}\delta$: and in the line $\alpha\,\delta$ take $\mathrm{Q}\delta$, and $\mathrm{P}\delta$ equal to the chords DN, DL, and through the points P, Q, draw the lines $\mathrm{P}p$, $\mathrm{Q}q$. parallel to $\alpha\mathrm{M}$, meeting MB in p and q.

Then 'tis evident in the firſt place, that lk will be $=$ AL, Ay $=$ AL, and Dy $=$ DN, and conſequently yN will be $=$ the difference of AN, AL, and Ly $=$ the difference of DL, DN, and therefore $=$ PQ. Further, becauſe the ordinates lk, np, are infinitely near to each other, the mixtilinear area $l\,k\,p\,n$ will be equal to the rectangle $l\,k\,r\,n$; and, becauſe by *Corol.* 7, we have yN : LN :: LD : AD, and for the ſame reaſon, LN : Ly :: AD : AL, it will follow, *ex æquo*, that yN : Ly :: LD : AL; conſequently LN will be $= \frac{\mathrm{AD} \times y\mathrm{N}}{\mathrm{LD}}$, and L$y$ $=$ $\frac{\mathrm{AL} \times y\mathrm{N}}{\mathrm{LD}}$. Therefore the mixtilinear area $lkpn$ is

$$= ln \times lk = \mathrm{LN} \times \mathrm{AL} = \frac{\mathrm{AD} \times y\mathrm{N}}{\mathrm{LD}} \times \mathrm{AL}$$

$$= \mathrm{AD} \times \frac{\mathrm{AL} \times y\mathrm{N}}{\mathrm{LD}} = \mathrm{AD} \times \mathrm{L}y = \mathrm{P}p \times \mathrm{PQ}.$$

And in the ſame manner it may be ſhewn that, if the area $a\,p\,n\,a$ be divided into an infinite number of infinitely ſmall mixtilinear areas, ſuch as $l\,k\,p\,n$, and the rectangle $\alpha\mathrm{M}q\mathrm{Q}$ be divided into the ſame number of infinitely ſmall rectangles, ſuch as PpqQ, correſponding to the portions of the area *a p n a*, each of the portions into which the former area is divided will be equal to the correſponding portion of the latter area: therefore the whole former area is equal to the whole latter area; that is, the area *a p n a* is $= \alpha M \times \alpha Q = \alpha M \times \overline{\alpha\delta - Q\delta} = AD \times \overline{AD - DN}$. QED.

Note, In the ſame manner 'tis evident, that if in the ſemicircle ABD in *Fig*. 93. we take the arch AH, which is greater than a quadrant, the area of the portion *a p k* H *b a* of the figure of chords in *Fig*. 94, correſponding to the arch AH, or whoſe baſe *a b* is equal to the arch AH, will be equal to $AD \times \overline{AD - \text{chord } DH}$, or to the rectangle under the diameter AD, and the exceſs of that diameter above the chord of the arc DH, or of the ſupplement of the arc AH, to which the baſe *a b* is equal, to a ſemicircle. For, by *Coroll*. 7, the proportion of the infinitely ſmall increment of the arc to the contemporary increment of the chord, upon which the foregoing reaſonings concerning the magnitude of the areas of this figure were founded, is the ſame, whether the arc be greater or leſs than a quadrant, as long as it is leſs than a ſemicircle: and conſequently the foregoing demonſtration will be equally true in both caſes.

324. Hence 'tis evident that the whole figure of chords *ap*KT*da*, correſponding to the ſemicircle ABD, is equal to the ſquare of the diameter AD.

325. The area of this figure may alſo be derived from that of the figure of ſines. For if the baſe *ab* of the portion *ap*H*ba* of the figure of

chords *apHTda* (*Fig.* 94.) be double of the baſe *an* of the portion *apna* of the figure of ſines *apKda* (*Fig.* 86.) the arch AH, to which the baſe *ab* is equal, will be double of the arch AN, to which the baſe *an* is equal; therefore the chord of the arch AH will be double of the ſine of the arch AN, and conſequently the ordinate *b*H, wich is equal to the chord AH, will be double of the ordinate *np* (*Fig.* 86.) which is equal to the ſine NS. And for the ſame reaſon 'tis evident that, if we divide the baſe *ab* (in *Fig.* 94.) into an infinite number of infinitely ſmall and equal parts, and draw the correſponding ordinates, and alſo divide the baſe *an* (*Fig.* 86.) into the ſame number of infinitely ſmall and equal parts, and draw the correſponding ordinates, every ordinate in the former figure will be double of the correſponding ordinate in the latter figure. But 'tis evident, that every abſciſs of the baſe *ab* in the former figure will be double of the correſponding abſciſs of the baſe *an* in the latter figure; and conſequently the difference of every two contiguous abſciſſes in the former figure will be double of the difference of the two correſpondent abſciſſes in the latter figure; therefore each of the mixtilinear areas into which the former figure is divided by its ordinates will be double both in height and in breadth of the correſponding mixtilinear area in the latter figure, and therefore will upon the whole be quadruple of it. Conſequently the whole area *apHba* in *Fig.* 94.) will be quadruple of the whole (area *apna* (in *Fig.* 86.) QED.

326. This determination agrees with that given above in *Art.* 309. For, ſince the arch AH is double of AN, it follows that its ſupplement DH to a ſemi-circle will be double of the complement BN of AN to a quadrant. Therefore the chord DH is equal to twice the ſine of BN, or to 2 CS. Now the area of the portion *apna* of the figure of ſines *ap*K*da* (*Fig.* 86.) is, by *Corol.* 11, $= CA \times AS$. Therefore, by this laſt determination of the figure of chords, the area of the portion *ap*H*ba* of the figure of chords *ap*KT*da*(*Fig.* 94.) is $= 4\,CA \times AS = 2CA \times 2AS = AD \times 2AS = AD \times 2 \times \overline{CA - CS} = AD \times \overline{2CA - 2CS} = AD \times \overline{AD - DH}$, which is the value of this area found in article 323.

CONCLUSION.

Containing an account of the moſt remarkable ſerieſes relating to circular arches and logarithms.

Of the ſerieſes for expreſſing a circular arc in powers of its ſine, chord, verſed ſine, tangent, and ſecant.

327. THOSE who are acquainted with the common methods of calculating by fluxions will eaſily be able to derive from the foregoing, or 30th propoſition, and its 2d, 4th, 6th, and 7th *Corollaries*, all the ſerieſes for expreſſing the relations between a circular arc and its ſine, chord, verſed ſine, tangent, and ſecant. Thus, to find the ſeries for expreſſing the arc in powers of the tangent, we need only put *a* for the arc, *t* for the tangent, and *r* for the radius of the circle, and $\dot{a}$ and $\dot{t}$ for the fluxions, or infinitely ſmall contemporary increments, of the arc and tangent, and by *Prop.* 30. we ſhall have $\dot{t} : \dot{a} :: rr+tt : rr$; whence $\dot{a}$ is $= \frac{rr\dot{t}}{rr+tt} = \dot{t} - \frac{t^2\dot{t}}{rr} + \frac{t^4\dot{t}}{r^4} - \frac{t^6\dot{t}}{r^6} + \frac{t^8\dot{t}}{r^8} - \frac{t^{10}\dot{t}}{r^{10}} +$ &c. *ad infinitum*, and conſequently, taking the fluents on both

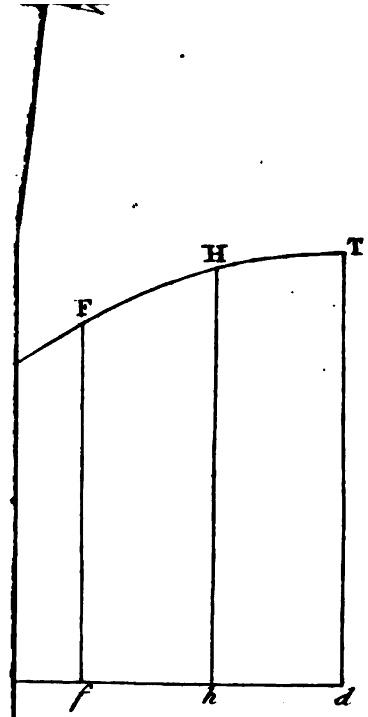
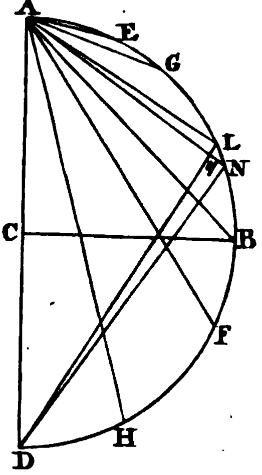


Fig. 93.



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both ſides, we ſhall have the arc a equal to the ſeries $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \&c$; which ſeries converges, and therefore exhibits the true value of the arc, as long as the tangent t is not greater than the radius, or as long as the arc is not greater than 45°.

SCHOLIUM.

Mr. James Gregory's *quadrature of the circle; and a hint towards deriving from thence a demonſtration of the impoſſibility of ſquaring the circle with perfect accuracy, if ſuch a quadrature be really impoſſible.*

328. If the arc is exactly 45°, we ſhall have $t =$ and conſequently the foregoing ſeries will be $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \&c.$ and therefore if r, or the radius of the circle, be called 1, the value of an arc of 45°, or of the eighth part of the whole circumference of the circle, will be equal to the ſeries $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c.$ *ad infinitum*, which is the ſeries which was given by Mr. James Gregory and Mr. Leibnitz for finding the quadrature of the circle, or exhibiting in numbers the proportion of the circumference of a circle to its diameter. But this purpoſe it was but ill fitted to anſwer, on account of the prodigious ſlowneſs with which it converges, which is ſo great that Sir

Iſaac Newton obſerves concerning this, and another ſeries almoſt as ſlow, that, to get the value of an arc of 45°, or of a quadrant, to twenty decimal places of figures, there would be occaſion for no leſs than five thouſand millions of terms of it, to compute which would take up above a thouſand years. [See his ſecond letter to Mr. Oldenburgh, dated Oct. 24, 1676, in the *Commercium Epiſtolicum*, page 159.] It muſt however be obſerved, that though this ſeries itſelf is of too ſlow a convergency to be made uſe of in calculation, yet it may be converted into another that converges more ſwiftly by the ingenious method invented by the very learned Mr. Stirling, which the reader may ſee explained in that author's excellent treatiſe, *De Summatione Serierum*, where he will alſo find, in page 32, an application of this method to the caſe in queſtion, or a quadrature of the circle perform'd to ten places of figures by means of a ſeries deriv'd from the ſeries $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} +$ &c. But the principal merit of this ſeries is the beautiful ſimplicity of its terms; which makes it alſo more peculiarly proper than any other ſeries of the like kind for being made the ground of a demonſtration of the incommenſurability of the circumference of a circle to its diameter, if ſuch incommenſurability (which, though often aſſerted, has never yet, as far as I have been able, after a good deal of ſearching and inquiry, to diſcover, been demonſtrated) be really demonſtrable. For if it can be ſhewn that no fraction whatſoever, vulgar or decimal, conſiſting of any finite number of figures, can accurately expreſs the value of this ſeries; it will follow, that the arc of 45° and the radius, and conſequently the circumference and diameter of a circle, are incommenſurable to each other; and this inquiry can, as I imagine, be made much more eaſily of ſuch a ſeries as this, that conſiſts of very ſimple fractions without any mixture at all of ſurd quantities, than of the ſeries $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}\times 3\times 3} + \frac{1}{\sqrt{3}\times 9\times 5} \quad \frac{-1}{\sqrt{3}\times 27\times 7} + \frac{1}{\sqrt{3}\times 81\times 9} \quad \frac{-1}{\sqrt{3}\times 243\times 11}$ &c. which Dr. Halley uſed for the quadrature of the circle, from the tangent of 30°, or $\frac{1}{\sqrt{3}}$, or of any other ſeries that involves in it one or more ſurd quantities.

Note, the foregoing ſeries $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}$ &c. is evidently equal to $\frac{2}{3}+\frac{2}{35}+\frac{2}{99}+$ &c. or $\frac{2}{2^2-1}+\frac{2}{6^2-1}+\frac{2}{10^2-1}+\frac{2}{14^2-1}+\frac{2}{18^2-1}+\frac{2}{22^2-1}+$ &c. and ſometimes is conſidered under that form.

Of the ſeries, for finding the arc frm its ſine.

329. Again, to find the ſeries for expreſſing the arc in powers of the ſine, put a for the arc, s for the ſine, and r for the radius, and $\dot{a}$ and $\dot{s}$ for the fluxions, or infinitely ſmall contemporary increments of the arc and ſine; and by *Coroll.* 2, we ſhall have $\dot{a} : \dot{s} :: r : \sqrt{rr-ss}$; whence $\dot{a}$ is $= \frac{r\dot{s}}{\sqrt{rr-ss}} = \dot{s} + \frac{s^2\dot{s}}{2r^2} + \frac{3s^4\dot{s}}{2.4.r^4} + \frac{3.5.s^6\dot{s}}{2.64..r^6} + \frac{3.5.7.s^8\dot{s}}{2.4.6.8.r^8} + \frac{3.5.7.9.s^{10}\dot{s}}{2.4.6.8.10.r^{10}} +$ &c. and conſequently, taking the fluents on both ſides, we ſhall have $a = s + \frac{s^3}{2.3.r^2} + \frac{3s^5}{2.4.5r^4} + \frac{3.5.s^7}{2.4.6.7.r^6} + \frac{3.5.7.s^9}{2.4.6.8.9.r^3} + \frac{3.5.7.9.s^{11}}{2.4.6.8.10.11.r^{10}}$ + &c. *ad infinitum*, the terms of which ſeries are generated from each other by a ſucceſſive multiplication into the fractions $\frac{1.1.s^2}{2.3.r^2}$, $\frac{3.3.s^2}{4.5.r^2}$, $\frac{5.5.s^2}{6.7.r^2}$, $\frac{7.7.s^2}{8.9.r^2}$, $\frac{9.9\,s^2}{10.11.r^2}$, &c. of which the ſquares of the odd numbers in their natural order multiplied into the ſquare of s compoſe the numerators, and the products of the next two natural numbers multiplied into the ſquare of r compoſe the denominators.

An eaſy quadrature of the circle.

If the arc a be 30°, the ſine s will be equal to half the radius; and therefore, if the radius be put $=1$, we ſhall have a, or the arc of 30°,

$$=\frac{1}{2}+\frac{1}{2.3.8}+\frac{3}{2.4.5.32}+\frac{3.5.}{2.4.6.7.128}+\frac{3.5.7.}{2.4.6.8.9.512}+\frac{3.5.7.9.}{2.4.6.8.10.11.2048}+\&c.$$

which ſeries converges, as is evident upon inſpection, with ſufficient expedition to afford a very eaſy method of performing the quadrature of the circle.

SCHOLIUM.

Of the excellency of thoſe methods of ſquaring the circle above that of Archimedes *or* Van Ceulen.

330. Indeed it is ſurprizing to conſider how infinitely eaſier and more expeditious both this and Dr. Halley's method abovementioned of ſquaring the circle, from the tangent of 30°, are than that which Archimedes and Van Ceulen made uſe of for the ſame purpoſe. For a hundred terms of either of theſe ſerieſes will give the length of an arc of 30°, and conſequently of the ſemicircumference, to more places of figures than Van Ceulen has computed it to; and yet, I dare ſay, a hundred terms might be computed in the ſpace of a very few hours by a ſkilful arithmetician, whereas Van Ceulen's computation was the produce of a prodigious deal of time and labour. Mr. Sharp has computed the circumference of a circle by Dr. Halley's method to no leſs than 72 places of figures, which is exactly double the number to which Van Ceulen has carried his calculation,

as may be ſeen in Sherwin's mathematical tables, pages 56, 57, 58, 59, in which the whole proceſs is ſet down, as well as ſeveral other quadratures of the circle performed by means of theſe ſerieſes, which the curious reader may conſult.

But the very beſt method of all of performing this quadrature is without queſtion that of the ingenious Mr. Machin, which is publiſhed in the appendix to my diſſertation on the uſe of the negative ſign in algebra.

Note, If the coefficients of the terms of the foregoing ſeries $s + \frac{s^3}{2.3.r^2} + \frac{3.s^5}{2.4.5.r^4} + \frac{3.5.s^7}{2.4.6.7.r^6} +$ &c. be properly reduced into ſingle fractions of the ſimpleſt kinds (by multiplying their factors together, and throwing out thoſe that are common to both the numerators and denominators) as they ought to be before the ſum of any number of them is computed, that ſeries will be as follows, to wit, $s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4} + \frac{5s^7}{112r^6} + \frac{35s^9}{1152r^8} + \frac{63s^{11}}{2816r^{10}} + \frac{231s^{13}}{13,312r^{12}} + \frac{143s^{15}}{10,240r^{14}} + \frac{6435,s^{17}}{557,056r^{16}} + \frac{206,635,s^{19}}{21,168,128r^{18}} + \frac{785,213,s^{21}}{93,585,408r^{20}} + \frac{1,499,043s^{23}}{204,996,608r^{22}} +$ &c.

Of

Of the ſeries for finding the arc from its chord.

331. 3*dly*, To find the arc in powers of the chord, put a for the arc, c for the chord, and d for the diameter of the circles, and $\dot{a}$ and $\dot{c}$ for the fluxions, or contemporary increments, of the arc and chord; and, by *Coroll.* 7, we ſhall have $\dot{a} : \dot{c} :: d : \sqrt{dd-cc}$; whence $\dot{a}$ is $= \frac{d\dot{c}}{\sqrt{dd-cc}}$ $= \dot{c} + \frac{c^2\dot{c}}{2d^2} + \frac{3c^4\dot{c}}{2.4.d^4} + \frac{3.5.c^6\dot{c}}{2.4.6.d^6} + \frac{3.5.7.c^8\dot{c}}{2.4.6.8.d^8} + \frac{3.5.7.9.c^{10}\dot{c}}{2.4.6.8.10.d^{10}} +$ &c. and conſequently, taking the fluents on both ſides, we ſhall have a, or the arc, $= c + \frac{c^3}{2.3.d^2} + \frac{3c^5}{2.4.5.d^4} + \frac{3.5.c^7}{2.4.6.7.d^6} + \frac{3.5.7.c^9}{2.4.6.8.9.d^8} + \frac{3.5.7.9.c^{11}}{2.4.6.8.10.11.d^{10}} +$ &c. in which the coefficients of the terms are exactly the ſame as in the foregoing ſeries derived from the ſine.

If therefore the terms of this ſeries be reduc'd to their ſimpleſt denominations, it will be as follows, to wit, $c + \frac{c^3}{6d^2} + \frac{3c^5}{40d^4} + \frac{5c^7}{112d^6} + \frac{35c^9}{1152d^8} + \frac{63c^{11}}{2816d^{10}} + \frac{231c^{13}}{13,312d^{12}} + \frac{143c^{15}}{10,240d^{14}} + \frac{6435c^{17}}{557,056d^{16}} + \frac{206,635c^{19}}{21,168,128d^{18}} + \frac{785,213c^{21}}{93,585,408d^{20}} + \frac{1,499,643c^{23}}{204,996,608d^{22}} +$ &c. as in the former ſeries.

Of the ſeries for finding the arc from the verſed ſine.

332. *4thly*, To find the arc in powers of the verſed ſine, put a for the arc, v for the verſed ſine, and d for the diameter, and $\dot a$ and $\dot v$ for the fluxions, or infinitely ſmall contemporary increments of the arc and verſed ſine, and, by *Coroll.* 6, we ſhall have $\dot a : \dot v :: \frac{d}{2} : \sqrt{dv - vv}$; whence $\dot a$ is $= \frac{d\dot v}{2\sqrt{dv-vv}} = \frac{d^{\frac12}v^{-\frac12}\dot v}{2} + \frac{v^{\frac12}\dot v}{2.2.d^{\frac12}} + \frac{3v^{\frac32}\dot v}{2.2.4.d^{\frac32}} + \frac{3.5.v^{\frac52}\dot v}{2.2.4.6.d^{\frac52}} + \frac{3.5.7.v^{\frac72}\dot v}{2.2.4.6.8.d^{\frac72}} + \frac{3.5.7.9.v^{\frac92}\dot v}{2.2.4.6.8.10.d^{\frac92}} + \frac{3.5.7.9.11.v^{\frac{11}{2}}\dot v}{2.2.4.6.8.10.12.d^{\frac{11}{2}}} +$ &c. and conſequently, taking the fluents on both ſides, we ſhall have a, or the arc, $= d^{\frac12}v^{\frac12} + \frac{v^{\frac32}}{2.3.d^{\frac12}} + \frac{3.v^{\frac52}}{2.4.5.d^{\frac32}} + \frac{3.5.v^{\frac72}}{2.4.6.7.d^{\frac52}} + \frac{3.5.7.v^{\frac92}}{2.4.6.8.9.d^{\frac72}} + \frac{3.5.7.9.v^{\frac{11}{2}}}{2.4.6.8.10.11.d^{\frac92}} + \frac{3.5.7.9.11.v^{\frac{13}{2}}}{2.4.6.8.10.12.13.d^{\frac{11}{2}}} +$ &c. *ad infinitum*; or, if, for the ſake of avoiding the fractional indexes, we put $\sqrt{dv} = z$, we ſhall have a, or the arc, $= z + \frac{z^3}{2.3.dd} + \frac{3z^5}{2.4.5d^4} + \frac{3.5.z^7}{2.4.6.7.d^6} + \frac{3.5.7.z^9}{2.4.6.8.9.d^8} + \frac{3.5.7.9.z^{11}}{2.4.6.8.10.11.d^{10}} + \frac{3.5.7.9.11.z^{13}}{2.4.6.8.10.12.13.d^{12}} +$ &c. which ſeries converges,

verges, and conſequently repreſents the true value of the arc, as long as the arc is of any magnitude leſs than a ſemicircle.

This ſeries, 'tis evident, has the ſame numeral coefficients with the two foregoing ſerieſes derived from the ſine and chord, and therefore, if its terms be properly reduced, will be as follows, to wit, $z + \frac{z^3}{6,dd} + \frac{3z^5}{40,d^4} + \frac{5z^7}{112,d^5} + \frac{35z^9}{1152,d^8} + \frac{63z^{11}}{2816,d^{10}} + \frac{231z^{13}}{13,312,d^{12}} + \frac{143z^{15}}{10,240,d^{14}} + \frac{6435z^{17}}{557,056,d^{16}} + \frac{206,635,z^{19}}{21,168,128,d^{18}} + \frac{785,213,z^{21}}{93,585,408^{20}} + \frac{1,499,043,z^{23}}{204,996,608,d^{22}} +$ &c. Indeed when expreſſed in this manner by putting $z = \sqrt{dv}$ this ſeries exactly coincides or becomes the very ſame with the laſt ſeries derived from the chord c; becauſe z being $= \sqrt{dv}$, or being a mean proportional between d and v, or the diameter and verſed ſine, muſt be equal to the chord c.

Of the ſeries for finding the arc from the variable part of the ſecant.

333. *5thly*, To find the arc in powers of the variable part of the ſecant, or of its exceſs above the radius, put a for the arc, x for the exceſs of the ſecant above the radius, r for the radius, and d for the diameter of the circle, and $\dot{a}$ and $\dot{x}$ for the fluxions, or infinitely ſmall contemporary increments of the arc and ſecant; and,

by *Coroll.* 4, we ſhall have $\dot x : \dot a :: \overline{r+x} \times \sqrt{2rx+xx} : rr$; whence $\dot a$ is $= \frac{rr\dot x}{\overline{r+x} \times \sqrt{2rx+xx}}$ $= \frac{x^{-\frac{1}{2}}\dot x}{2^{\frac{1}{2}} \times r^{-\frac{1}{2}}} + \frac{2.2.A+1}{2 \times 2^{\frac{1}{2}}} \times \frac{-x^{\frac{1}{2}}\dot x}{r^{\frac{1}{2}}} + \frac{2.4.B+3}{2.4 \times 2^{\frac{3}{2}}} \times \frac{x^{\frac{3}{2}}\dot x}{r^{\frac{3}{2}}} + \frac{2.6.C+3.}{2.4.6. \times 2^{\frac{5}{2}}} \times \frac{-x^{\frac{5}{2}}\dot x}{r^{\frac{5}{2}}} + \frac{2.8.D+3.5.7}{2.4.6.8. \times 2^{\frac{7}{2}}} \times \frac{x^{\frac{7}{2}}\dot x}{r^{\frac{7}{2}}}$ + *&c. ad infinitum*; in which the capital letters A,B,C, *&c.* repreſent the numerators of the coefficients of the terms as they ariſe, A being equal to 1 or the numerator of the coefficient of the firſt term, B to 2.2.A+1, or to 2×2×1+1, that is, to 4+1, or 5, and C to 2.4.B+3, or 2×4×5+3, or 40+3, or 43, and D to 2.6.C+3.5, or 2×6×43+15, or 516+15, or 531, and ſo of the reſt; and the additional parts of the numerators, to wit, 3. 3.5. 3.5.7, *&c.* are the products of the odd numbers taken in their natural order, and the denominators are the products of the even numbers taken in their natural order, and multiplied into the odd powers of the root of 2, taken alſo in their natural order. Taking therefore the fluents of theſe terms, we ſhall have 4, or the arc, $= \frac{x^{\frac{1}{2}}}{2^{-\frac{1}{2}}r^{-\frac{1}{2}}} + \frac{2.2.A+1}{2.3. \times 2^{\frac{1}{2}}} \times \frac{-x^{\frac{3}{2}}}{r^{\frac{1}{2}}} + \frac{2.4.B+3}{2.4.5. \times 2^{\frac{3}{2}}} \times \frac{x^{\frac{5}{2}}}{r^{\frac{3}{2}}} + \frac{2.6.C+3.5}{2.4.6.7. \times 2^{\frac{5}{2}}} \times \frac{-x^{\frac{7}{2}}}{r^{\frac{5}{2}}} + \frac{2.8.D+3.5.7}{2.4.6.8.9. \times 2^{\frac{7}{2}}} \times \frac{x^{\frac{9}{2}}}{r^{\frac{7}{2}}}$ + *&c.* or, if we ſubſtitute the diameter d in the room of $2r$, (by which ſubſtitution the ſeries will become ſomewhat ſimpler) we ſhall have $a = \frac{x^{\frac{1}{2}}}{d^{-\frac{1}{2}}} + \frac{2.2.A+1}{2.3} \times \frac{-x^{\frac{3}{2}}}{d^{\frac{1}{2}}} + \frac{2.4.B+3}{2.4.5.} \times \frac{x^{\frac{5}{2}}}{d^{\frac{3}{2}}} + \frac{2.6.C+3.5}{2.4.6.7} \times \frac{-x^{\frac{7}{2}}}{d^{\frac{5}{2}}} + \frac{2.8.D+3.5.7}{2.4.6.8.9.} \times \frac{x^{\frac{9}{2}}}{d^{\frac{7}{2}}} +$ &c. or, if, for the ſake of avoiding the fractional indexes, we put $\frac{x^{\frac{1}{2}}}{d^{-\frac{1}{2}}}$, or $x^{\frac{1}{2}} \times d^{\frac{1}{2}}$, or $\sqrt{x} \times \sqrt{d}$, or $\sqrt{dx}$, or the mean proportional between d and x, or between the diameter and the exceſs of the ſecant above the radius, $= y$, (which mean proportional, we may obſerve, will not be any known line in the circle, but will be ſomewhat leſs than the tangent, being accurately equal to a mean proportional between the tangent and a line equal to twice the tangent of half the arc) we ſhall have a, or the arc, $= y + \frac{2.2.A+1}{2.3} \times \frac{-y^3}{dd} + \frac{2.4.B+3}{2.4.5.} \times \frac{y^5}{d^4} + \frac{2.6\,C+3.5}{2.4.6.7} \times \frac{-y^7}{d^6} + \frac{2.8.D+3.5.7}{2.4.6.8.9} \times \frac{y^9}{d^8} +$ &c. *ad infinitum*; in which the law of the continuation of the terms is ſufficiently evident from the inſpection of the ſeries, and from what has been already ſaid to explain it. And it muſt be further obſerved, that this ſeries converges, and conſequently exhibits the true value of the arc, as long as x is not greater than r, or the ſecant is not greater than twice the radius, or as long as the arc is not greater than 60 degrees.

Of the ſerieſes for expreſſing the ſines, chords, verſed ſines, tangents, and ſecants of a circular arc in powers of the arc.

334. THESE are the ſeveral ſerieſes by which the arc of a circle may be derived from the ſeveral right lines that belong to it. It remains that we give the reverſes of theſe ſerieſes, or the ſerieſes by which theſe ſeveral right lines may be expreſs'd in powers of the arc; which may either be obtained ſeparately by reverting, as it is call'd, the ſeveral foregoing ſerieſes, or when one of them, as for inſtance, the ſeries for expreſſing the ſine in powers of the arc, is obtained by this method of reverſion, the others may be derived from it by the common operations of arithmetic and extraction of the ſquare root, which is much eaſier than repeating anew the method of reverſion.

Of the ſeries for expreſſing the ſine in powers of the arc.

335. To revert the ſeries $s+\frac{s^3}{2.3.r^2}+\frac{3s^5}{2.4.5.r^4}+$ *&c.* aſſume $s=$ to the ſeries $Aa+Ba^3+Ca^5+$ *&c. ad infinitum*; and we ſhall have the cube of this ſeries, or $A^3a^3+3A^2Ba^5+$*&c.*$=s^3$, and the fifth power of this ſeries, or A^5a^5+*&c.*$=s^5$, and ſo on of the following powers; and conſequently $a(=s+\frac{s^3}{2.3.r^2}+\frac{3s^5}{2.4.5.r^4}+$ *&c.*$)=$

As

$$\begin{aligned} &Aa + Ba^3 + Ca^5 + \&c. \\ &+\frac{A^3a^3}{2.3.r^2} + \frac{3A^2Ba^5}{2.3.r^2} + \&c. \\ &+\frac{3A^5a^5}{2.4.5.r^4} + \&c. \end{aligned}$$

whence it follows, (ſince this equation is true when a is infinitely ſmall as well as when it is of a finite magnitude) that Aa will be a, and $Ba^3 + \frac{A^3a^3}{2.3.r^2}$ will be $= 0$, and $Ca^5 + \frac{3A^2Ba^5}{2.3.r^2} + \frac{3A^5a^5}{2.4.5.r^4} = 0$, and in like manner the ſum of the terms involving every following power of a will be $= 0$; whence, it is evident, the coefficients A, B, C, &c. may be determined. Thus from the firſt of theſe equations, to wit, $Aa = a$, we have $A = 1$; and from the ſecond of theſe equations, to wit, $Ba^3 + \frac{A^3a^3}{2.3.r} = 0$, we have $B = \frac{-1}{2.3.r^2}$; and from the third equation we have $C \left(= -\frac{3A^2B}{2.3.r^2} - \frac{3A^5}{2.4.5.r^4} = \frac{-3}{2.3.r^2} \times \frac{-1}{2.3.r^2} - \frac{3}{2.4.5.r^4} = \frac{+1}{2.2.3.r^4} - \frac{3}{2.4.5r^4} = \frac{10}{2.3.4.5r^4} - \frac{9}{2.3.4.5.r^4}\right) = \frac{1}{2.3.4.5.r^4}$; and in like manner the following coefficients may be determined, and will be found to be $\frac{-1}{2.3.4.5.6.7.r^6}$, $\frac{+1}{2.3.4.5.6.7.8.9.r^8}$, $\frac{-1}{2.3.4.5.6.7.8.9.10.11.r^{10}}$, &c.

but it muſt be confeſs'd that the reſolution of the equations by which theſe coefficients are determined, becomes, after the firſt four or five terms, prodigiouſly laborious. It appears therefore that s, or the ſine of the arc a, is equal to the ſeries $a - \frac{a^3}{2.3r^2} + \frac{a^5}{2.3.4.5r^4} - \frac{a^7}{2.3.4.5.6.7r^5} + \frac{a^9}{2.3.4.5.6.7.8.9.r^8} - \frac{a^{11}}{2.3.4.5.6.7.8.9.10.11.r^{10}}$ &c. in which the law of continuation is very manifeſt.

If the coefficients of this ſeries are properly reduced, it will be as follows, to wit, $a - \frac{a^3}{6r^2} + \frac{a^5}{120r^4} - \frac{a^7}{5040r^5} + \frac{a^9}{362,880r^8} - \frac{a^{11}}{39,916,800r^{10}}$ &c. This ſeries, 'tis evident, converges ſo exceeding ſwiftly that a very few terms of it, ſeldom more than five or ſix, are ſufficient for moſt purpoſes of calculation.

Of the ſerieſes for expreſſing the coſine and verſed ſine in powers of the arc.

336. By ſquaring this ſeries, and ſubtracting its ſquare from the ſquare of the radius, and then extracting the ſquare root of the remainder, we ſhall obtain for the value of the coſine the following ſeries, to wit, $r - \frac{a^2}{2.r} + \frac{a^4}{2.3.4.r^3} - \frac{a^6}{2.34.5.6.r} + \frac{a_8}{2.3.4.5.6.7.8.r_7} - \frac{a^{10}}{2.3.4.5.6.7.8.9.10.r^9} +$ &c. which being ſubtracted from the radius,

leaves the verſed ſine $= \frac{a^2}{2r} - \frac{a^4}{2.3.4.r^3} + \frac{a^6}{2.3.4.5.6.r^5} - \frac{a^8}{2.3.4.5.6.7.8.r^7} + \frac{a^{10}}{2,3.4.5.6.7.8.9.10r^9}$ &c. or, if the terms of this ſeries be properly reduced, $\frac{a^2}{2r} - \frac{a^4}{24r^3} + \frac{a^5}{720r^5} - \frac{a^8}{40,320r^7} + \frac{a^{10}}{3,628,800r^9}$ &c.

Of the ſerieſes for expreſſing the ſecant and tangent in powers of the arc.

337. The ſeries for the coſine being found, that for the ſecant may be derived from it by dividing the ſquare of the radius by the former ſeries, and will be found to be as follows; $r + \frac{a^2}{2r} + \frac{5a^4}{2.3.4.r^3} + \frac{61a^6}{2.3.4.5.6.r^5} + \frac{1385a^8}{2.3.4.5.6.7.8.r^7} + \frac{50,521a^{10}}{2.3.4.5.6.7.8.9.10.r^9} +$ &c. And by ſquaring this laſt ſeries, and ſubtracting the ſquare of the radius from its ſquare, and then extracting the ſquare root of the remainder, we ſhall find the ſeries for the tangent to be as follows, to wit, $a + \frac{8a^3}{2.3.4.r^2} + \frac{96a^5}{2.3.4.5.6.r^4} + \frac{2176a^7}{2.3.4.5.6.7.8.r^6} + \frac{79360a^9}{2.3.4.5.6.7.8.9.10.r^5} +$ &c.

If theſe two ſerieſes be properly reduced, the former of them, or that for the ſecant, will be as follows, $r + \frac{a^2}{2r} + \frac{5a}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} + \frac{50,521a^{10}}{3,628,800r^9} +$ &c. and the ſeries for the tangent will be $a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} +$ &c.

Of the ſeries for expreſſing the chord in powers of the arc.

338. As to the ſeries for the chord, 'tis evident it muſt be exactly ſimilar to the ſeries for the ſine, as the two ſerieſes for expreſſing the arc in powers of the chord and ſine are exactly alike. Therefore c, or the chord of any arc a in a circle whoſe diameter is d, is $= a - \frac{a^3}{2.3.d^2} + \frac{a^5}{2.3.4.5.d^4} - \frac{a^7}{2.3.4.5.6.7.d^6} + \frac{a^9}{2.3.4.5.6.7.8.9.d^8} - \frac{a^{11}}{2.3.4.5.6.7.8.9.10.11d^{10}} +$ &c. or, if the terms be properly reduced, $a - \frac{a^3}{6d^2} + \frac{a^5}{120d^4} - \frac{a^7}{5040d^6} + \frac{a^9}{362,880d^8} - \frac{a^{11}}{39,916,800d^{10}} +$ &c. as in the ſeries for the ſine.

Note, Thoſe who are deſirous of ſeeing thoſe ſerieſes continued to a greater number of terms than have been here ſet down, may have their curioſity ſatisfied by conſulting Mr. Sharp's method of conſtructing natural ſines and tangents, which is bound up with Sherwin's mathematical tables. See the 3d edit. 1741, page 45.

Of the ſeries for finding the ſine of a multiple arc from the ſine of the ſimple arc.

339. THERE is another ſeries which it will be proper to mention on this occaſion. This is the ſeries for expreſſing the ſine of any given multiple arc in powers of the ſine of the ſimple arc, and may be derived from the ſeries $s + \frac{s^3}{2.3r^2} + \frac{3s^5}{2.3.4.5.r^4} +$ &c. by the ſame method of reverſion by which the ſeries $a - \frac{a^3}{2.3.r^2} + \frac{a^5}{2.3.4.5.r^4} +$ &c. was derived from that ſeries.

For let the multiple arc contain the ſimple arc n times; and let r be the radius of the circle, s the ſine of the ſimple arc, and x the ſine of the multiple arc. And we ſhall have the ſimple arc equal to $s + \frac{s^3}{2.3.r^2} + \frac{3.s^5}{2.4.5r^4} +$ &c. and the multiple arc equal to $x + \frac{x^3}{2.3.r^2} + \frac{3x^5}{2.4.5r^4} +$ &c. and conſequently n times the former ſeries will be equal to the latter ſeries; that is, $ns + \frac{ns^3}{2.3.r^2} + \frac{3ns^5}{2.4.5r^4} +$ &c. is $= x + \frac{x^3}{2.3.r^2}$ +

$+\frac{3x^5}{2.4.5.r^4}+$ &c. or $x+\frac{x^3}{2.3.r^2}+\frac{3^5}{2.4.5.r^4}+$ &c. $-ns-\frac{ns^3}{2.3.r^2}-\frac{3ns^5}{2.4.5.r^4}-$ &c. is $=0$.

Aſſume therefore $x=As+Bs^3+Cs^5+$ &c. and we ſhall have

$$\left.\begin{array}{l} As+Bs^3+Cs^5+\&c.\\ \quad+\frac{A^3s^3}{2.3.r^2}+3\frac{A^2Bs^5}{2.3.r^2}+\&c.\\ \qquad\qquad+\frac{3A^5s^5}{2.4.5.r^4}+\&c.\\ -ns-\frac{ns^3}{2.3.r^2}-\frac{3ns^5}{2.4.5.r^4}-\&c.\end{array}\right\}=0,$$ whence we have the following equations for determining the values of the coefficients A, B, C, &c. to wit, 1*ſt*, $A-n=0$; whence A is $=n$.

2*dly*, $B+\frac{A^3}{2.3.r^2}-\frac{n}{2.3.r^2}=0$; whence B is $=\frac{n-n^3}{2.3.r^2}$, or $n\times\frac{1-n^2}{2.3.r^2}$.

And 3*dly*, $C+\frac{3A^2B}{2.3.r^2}+\frac{3A^5}{2.4.5.r^4}-\frac{3n}{2.4.5.r^4}=0$; whence C is $=\frac{9n-10n^3+n^5}{2.3.4.5.r^4}$, or $n\times\frac{1-n^2}{2.3.r^2}\times\frac{9-n^2}{4.5.r^2}$: And in the ſame manner may be determined all the following coefficients D, E, F, &c.

Therefore x, or the ſine of the multiple arc, is $=ns+n\times\frac{1-nn}{2.3.r^2}\times s^3+n\times\frac{1-nn}{2.3.r}\times$

9—nn

$\frac{9-nn}{4.5.r^2} \times s^5 +$ &c. or, putting A, B, C, D, &c. for the values of the ſeveral terms of this ſeries reſpectively as they ariſe, we ſhall have

$$x = ns + \frac{1-nn}{2.3.r^2} \times As + \frac{9-nn}{4.5.r^2} \times Bs^2 + \frac{25-nn}{6.7.r^2} \times Cs^2 + \frac{49-nn}{8.9.r^2} \times Ds^2 + \frac{81-nn}{10.11.r^2} \times Es^2 + \frac{121-nn}{12.13.r^2} \times Fs^2 + \&c.$$

in which ſeries the numbers 1, 9, 25, 49, 81, 121, &c. are the ſquares of the odd numbers, 1, 3, 5, 7, 9, 11, &c. taken in their natural order.

SCHOLIUM.

340. This ſeries was invented by Sir Iſaac Newton, and communicated by him to Mr. Oldenburgh in a letter dated June 13, 1676, which may be ſeen in the Commercium Epiſtolicum, page 136. And the great excellence of it, or the advantage it has over the method of finding the ſines of multiple arcs delivered in *Prop.* 26, conſiſts in its enabling us to determine the ſine of the propoſed multiple arc at once, or *per ſaltum*, without previouſly determining the ſines of the intermediate multiple arcs, which is neceſſary in the other method.

If *n* is an odd number, or the greater arc is an odd multiple of the leſſer, 'tis evident the foregoing ſeries will break off, and conſiſt of a finite number of terms, as it ought to do according to the concluſions in *Prop.* 26. For in this caſe nn will in ſome term or other of the ſeries be equal to the number from which it is ſubtracted, which will make that term, and conſequently all the following terms which are generated from it by multiplication, become equal to o. But for the ſame reaſon, if the greater arc is an even multiple of the leſſer arc, or n is an even number, the ſeries will never break off, but will conſiſt of an infinite number of terms, agreeably likewiſe to the determinations of *Prop.* 26.

Thoſe who will try this ſeries in a few inſtances will eaſily perceive its agreement with the determinations in *Prop.* 26. But this, for brevity's ſake, I ſhall here omit doing.

A like ſeries for the tangent of a multiple arc, and Mr. John Bernouilli's *expreſſion for determining the ſame from the tangent of the ſimple arc.*

341. A like ſeries may be found for the tangent of a multiple arc. For if t be put for the tangent of the ſimple arc, and x for the tangent of the multiple arc that contains the former arc n times, it will be found by a proceſs ſimilar to the foregoing that x will be $= nt + \frac{n^3 - n}{3 r^2} \times t^3 + \frac{2n^5 - 5n^3 + 3n}{15 r^4} \times t^5 + \frac{17n^7 - 70n^5 + 98n^3 - 45n}{315 r^6} \times t^7 +$ &c. But as

the law of the continuation of the terms in this ſeries is not very eaſy to diſcover, it will be better to make uſe of the expreſſion invented by Mr. John Bernouilli for that purpoſe, by which, if the radius be call'd 1, we ſhall have $x =$

$$\frac{\frac{n}{1}t - \frac{n \times \overline{n-1} \times \overline{n-2}}{1.2.3} \times t^3 \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{1.2.3.4.5} \times t^5 + \&c.}{1 - \frac{n \times \overline{n-1}}{1.2} \times t^2 + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1.2.3.4} \times t^4 + \&c.}$$

See the works of John Bernouilli, edit. Genev. vol. II. page 532.

Of the ſerieſes for expreſſing the areas of the figures of tangents and ſecants.

342. By the help of the foregoing, or 10th propoſition, and its corollaries, we may alſo find ſerieſes for expreſſing the values of any given portions of the areas of the figures of tangents and ſecants. Thus, if the radius AC (*Fig.* 78) of the circle from which the given figures of tangents and ſecants are derived be called r, and the ſine SN of the arc AN that is equal to the baſes *an*, *an*, of the portions *apna* and AD*dna*, (in *Fig.* 79 and 80) of thoſe figures, is put $= s$, the area of the portion *apna* of the figure of tangents will be $= \frac{ss}{2} + \frac{s^4}{4rr} + \frac{s^6}{6r^4} + \frac{s^8}{8r^5} + \frac{s^{10}}{10r^3} + \frac{s^{12}}{12r^{10}} + \frac{s^{14}}{14r^{12}} +$ &c. *ad infinitum*, and the area

of the portion *aDδna* of the figure of ſecants will be $= rs + \frac{s^3}{3r} + \frac{s^5}{5r^3} + \frac{s^7}{7r^5} + \frac{s^9}{9r^7} + \frac{s^{11}}{11r^9} + \frac{s^{13}}{13r^{11}} +$ &c. *ad infinitum*, in both which ſerieſes the law of conſtruction is very manifeſt.

The manner of finding theſe ſerieſes is this. The fluxion of the area *apna* of the figure of tangents is = the mixtilinear area *klnp* = the rectangle *klnq* = AM x LN $= \frac{rs}{\sqrt{rr-ss}}$ x LN = (by *Coroll.* 2) $\frac{rs}{\sqrt{rr-ss}} \times \frac{r\dot{s}}{\sqrt{rr-ss}} = \frac{rrs\dot{s}}{rr-ss} = s\dot{s} + \frac{s^3\dot{s}}{rr} + \frac{s^5\dot{s}}{r^4} + \frac{s^7\dot{s}}{r^6} +$ &c. and conſequently, taking the fluents on both ſides, the area *apna* will be $= ss + \frac{s^4}{4rr} + \frac{s^6}{6r^4} + \frac{s^8}{8r^6} +$ &c. And in like manner, the fluxion of the area *aDδna* of the figure of ſecants is = the mixtilinear area *γlnδ* = the rectangle *γlnp* = CM x LN $= \frac{rr}{\sqrt{rr-ss}}$ x LN $= \frac{rr}{\sqrt{rr-ss}} \times \frac{r\dot{s}}{\sqrt{rr-ss}} = \frac{r^3\dot{s}}{rr-ss} = r\dot{s} + \frac{s^2\dot{s}}{r} + \frac{s^4\dot{s}}{r^3} + \frac{s^6\dot{s}}{r^5} +$ &c. and conſequently, taking the fluents on both ſides, the area *aDδna* will be $= rs + \frac{s^3}{3r} + \frac{s^5}{5r} + \frac{s^7}{7r^5} +$ &c.

Note, Theſe ſerieſes were firſt invented by Dr. Wallis, and publiſhed in the Philoſophical Tranſactions, and may alſo be ſeen in the Miſcellanea Curioſa, vol. II. page 9.

Conſequences deduced from the two foregoing ſerieſes.

343. Hence we may deduce the following conſequences.

In the firſt place, it follows from *art.* 276, that the ſeries $\frac{ss}{2} + \frac{s^4}{4r^2} + \frac{s^6}{6r^4} + \frac{s^3}{8r^6} + \frac{s^{10}}{10r^8} + \frac{s^{12}}{12r^{10}} + \frac{s^{14}}{14r^{12}} +$ &c. for the area of the figure of tangents, is equal to the logarithm of the ratio of the radius r of the circle to the coſine $\sqrt{rr\ ss}$ of the arc of which s is the ſine, in any hyperbola, or ſyſtem of logarithms, of which rr, or the ſquare of the radius, is the parallelogram, or modulus.

2dly, It follows alſo from the ſame article that the ſeries $rs + \frac{s^3}{3r} + \frac{s^5}{5r^4} + \frac{s^7}{7r^6} + \frac{s^9}{9r^6} + \frac{s^{11}}{11r^{10}} + \frac{s^{13}}{13r^{12}} +$ &c. for the area of the figure of ſecants, is equal to the logarithm of the ratio of the ſum of the radius of the circle and the ſine s to the coſine $\sqrt{rr - ss}$ in an hyperbola, or ſyſtem of logarithms, of which rr, or the ſquare of the radius, is the parallelogram, or modulus.

And therefore 3dly, If we ſubtract the former of theſe ſerieſes from the latter, the ſeries $rs - \frac{ss}{2} + \frac{s^3}{3r} - \frac{s^4}{4r^2} + \frac{s^5}{5r^3} - \frac{s^6}{6r^4} + \frac{s^7}{7r} - \frac{s^8}{8r^6} +$

s^9

$\frac{s^9}{9r^7} - \frac{s^{10}}{10r^8} + \frac{s^{11}}{11r^9} - \frac{s^{12}}{12r^{10}} +$ &c. thence obtained, will be equal to the logarithm of the ratio of the ſum of the radius r and ſine s to the radius r in the ſame ſyſtem of logarithms, or hyperbola, whereof rr is the parallelogram.

4thly, If s is $= r$, or the arc becomes equal to a quadrant, we ſhall have $r + s = 2r$, and the ratio of $r + s$ to r will in this caſe be equal to the ratio of 2 to 1; and the laſt mentioned ſeries will in this caſe become $= rr - \frac{rr}{2} + \frac{rr}{3} - \frac{rr}{4} + \frac{rr}{5} - \frac{rr}{6}$, &c. Therefore the ſeries $rr - \frac{rr}{2} + \frac{rr}{3} - \frac{rr}{4} + \frac{rr}{5} - \frac{rr}{6}$ &c. or $\frac{rr}{2} + \frac{rr}{3.4} + \frac{rr}{5.6} + \frac{rr}{7.8} + \frac{rr}{9.10} + \frac{rr}{11.12} + \frac{rr}{13.14} +$ &c. or $\frac{rr}{2} + \frac{rr}{12} + \frac{rr}{30} + \frac{rr}{56} + \frac{rr}{90} + \frac{rr}{132} + \frac{rr}{182} +$ &c. is the logarithm of the ratio of 2 to 1 in an hyperbola whoſe parallelogram is equal to rr.

And conſequently in the 5th place, if rr be call'd 1, the logarithm of the ratio of 2 to 1 will be $= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}$, &c. that is, Napier's logarithm of 2, or of the ratio of 2 to 1, is $=$ the ſeries $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} +$ &c.

or $\frac{1}{2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\frac{1}{9.10}+\frac{1}{11.12}+\frac{1}{13.14}$ + &c. or $\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}+\frac{1}{90}+\frac{1}{132}+\frac{1}{182}$ + &c.

6thly, Since the areas of the whole figures of tangents and ſecants, correſponding to an arc of 90°, or a quadrant, are each of them infinite in magnitude, as is ſhewn in *art.* 279, it follows, that the ſerieſes $\frac{ss}{2}+\frac{s^4}{4rr}+\frac{s^6}{6r^4}+\frac{s^8}{8r^6}+\frac{s^{10}}{10r^8}$ + &c. and $rs+\frac{s^3}{3r}+\frac{s^5}{5r^3}+\frac{s^7}{7r^5}+\frac{s^9}{9r^7}$ + &c. become each of them infinite in magnitude when they expreſs the areas of thoſe whole figures, or when s becomes $= r$; that is, the ſerieſes $\frac{rr}{2}+\frac{rr}{4}+\frac{rr}{6}+\frac{rr}{8}+\frac{rr}{10}$ + &c. and $rr+\frac{rr}{3}+\frac{rr}{5}+\frac{rr}{7}+\frac{rr}{9}$ + &c. are each of them infinite in magnitude; or, if rr be call'd 1, the ſerieſes $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$ + &c. and $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$ + &c. are each of them infinite in magnitude.

And thus it appears, in the 7th place, that, if from the ſeries $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$ + &c. which is infinite in magnitude, we ſubtract the ſeries $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+$ &c. which is alſo infinite in magnitude, the remainder thence obtained, or the ſeries $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+$ &c. or $\frac{1}{2}+\frac{1}{3\cdot4}+\frac{1}{5\cdot6}+\frac{1}{7\cdot8}+\frac{1}{9\cdot10}+$ &c. will be of a finite magnitude, and will be equal to Napier's logarithm of the ratio of 2 to 1; and, if from the ſame ſeries $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+$&c. which is infinite in magnitude, we ſubtract the ſeries $\frac{2}{3}+\frac{2}{7}+\frac{2}{11}+\frac{2}{15}+\frac{2}{19}+$ &c. which is alſo infinite in magnitude, the remainder thence obtained, or the ſeries $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}+$ &c. or $\frac{2}{3}+\frac{2}{5\cdot7}+\frac{2}{9\cdot11}+\frac{2}{13\cdot15}+\frac{2}{17\cdot19}+$ &c. will be of a finite magnitude, and will be equal to an arc of 45° in a circle whoſe radius is 1.

Scholium.

344. As to the figures of ſeries, chords, and verſed ſines, it has been already ſhewn how their areas may be obtained by multiplying the radius, or diameter, either into ſome of the right lines belonging to a circle, or into a circular arc; and we have already given a variety of ſe-

rieſes for determining the lengths of circular arcs: it is therefore needleſs to ſay any thing more of theſe areas on this occaſion.

Of the ſerieſes relating to logarithms derived from the contemplation of the hyperbola and logarithmic curve.

345. Let it be required in the firſt place to find an infinite ſeries for expreſſing the value of any given aſymptotic area of an hyperbola.

Let VA*d* (*Fig.* 70) be an hyperbola of any ſpecies whatſoever, equilateral or other; and let A be its vertex, C its center, and CT, CD its aſymptotes, and D*d* and E*e* two ordinates parallel to each other and to the aſymptote CT. Then, to find a ſeries for expreſſing the aſymptotic area D*de*E in powers of the abſciſs DE, we muſt proceed in this manner. Put *aa* for the parallelogram of the hyperbola, *b* for the line CD, and *x* for the line DE; and from any two points H, K in the line DE, taken infinitely near to each other, draw the ordinates H*b*, K*k* parallel to each other, and to the aſymptote CD; and conceive the whole area D*de*E to be divided in this manner by an infinite number of other ordinates, drawn parallel to CT, into ſuch little mixtilinear ſpaces as H*b*K*k*; and call HK, or the fluxion of the abſciſs DE, $\dot{x}$. Then 'tis evident, we ſhall have, by El. 6. 1. the ſpace H*bk*K, or the parallelogram HK*kp*, to the parallelogram CH*bω*, or the parallelogram of the hyperbola, as HK to

CH, or CD+DH; that is, HK*kp* : *aa* :: $\dot{x}$: $b+x$; whence HK*kp*, or the fluxion of the area D*de*E, is $= \frac{aa\dot{x}}{b+x} = \frac{aa\dot{x}}{b} - \frac{aax\dot{x}}{bb} + \frac{aax^2\dot{x}}{b^3} - \frac{aax^3\dot{x}}{b^4} + \frac{aax^4\dot{x}}{b^5} - \frac{aax^5\dot{x}}{b^6} + \frac{aax^6\dot{x}}{b^7} - \frac{aax^7\dot{x}}{b^8} +$ &c. and conſequently, taking the fluents on both ſides, the area D*de*E is $= \frac{aax}{b} - \frac{aaxx}{2b^2} + \frac{aax^3}{3b^3} - \frac{aax^4}{4b^4} + \frac{aax^5}{5b^5} - \frac{aax^6}{6b^6} + \frac{aax^7}{7b^7} - \frac{aax^8}{8b^8} \pm$ &c. or, in other words, the logarithm of the ratio of $b+x$ to b in an hyperbola whoſe parallelogram is $= aa$, is equal to the ſeries $\frac{aax}{b} - \frac{aaxx}{2b^2} + \frac{aax^3}{3b^3} - \frac{aax^4}{4b^4} + \frac{aax^5}{5b^5} - \frac{aax^6}{6b^6} + \frac{aax^7}{7b^7} - \frac{aax^8}{8b^8} +$ &c. which ſeries, 'tis evident, converges as long as x is leſs than, or equal to b, but no longer.

2*dly*, If we ſuppoſe the hyperbola to be equilateral, and take CD, or b, equal to a, the foregoing ſeries will be the logarithm of the ratio of $a+x$ to a, and will in this caſe become ſomewhat more ſimple, and will be as follows, $ax - \frac{xx}{2} + \frac{x^3}{3a} - \frac{x^4}{4aa} + \frac{x^5}{5a^3} - \frac{x^6}{6a^4} + \frac{x^7}{7a^5} - \frac{x^8}{8a^5} +$ &c. which, if r be ſubſtituted inſtead of a, and s inſtead of x, will coincide exactly with the ſeries $rs - \frac{ss}{2} + \frac{s^3}{3r} - \frac{s^4}{4rr} + \frac{s^5}{5r^3} - \frac{s^6}{6r^4} + \frac{s^7}{7r^5} - \frac{s^8}{8r^6} +$ &c. which was before found for this quantity, or for the logarithm of $r+s$ to r in an hyperbola whoſe parallelogram was $= rr$, in the third conſequence deduced from the ſerieſes for the figures of tangents and ſecants in art. 343.

3*dly*, And hence alſo it appears that the logarithm of the ratio of 2 to 1, in an hyperbola whoſe parallelogram is $= aa$, is $= aa - \frac{aa}{2} + \frac{aa}{3} - \frac{aa}{4} + \frac{aa}{5} - \frac{aa}{6} + \frac{aa}{7} - \frac{aa}{8} +$ &c. or $\frac{aa}{2} + \frac{aa}{3.4} + \frac{aa}{5.6} + \frac{aa}{7.8} +$ &c. for this is the value of the foregoing ſeries when x becomes $= a$. And this is agreeable to what has been already demonſtrated in art. 343.

4*thly*, And if aa is call'd 1, the laſt-found ſeries $aa - \frac{aa}{2} + \frac{aa}{3} - \frac{aa}{4} + \frac{aa}{5} - \frac{aa}{6} +$ &c. will be $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ &c. or $\frac{1}{2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} +$ &c. that is, Napier's logarithm of 2, or of the ratio of 2 to 1, is $= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} +$ &c. or $\frac{1}{2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} +$ &c. as has been ſhewn above in art. 343.

5*thly*, If it be required to find the logarithm of the ratio of b to $b-x$ in an hyperbola of any kind whoſe parallelogram is $= aa$, we ſhall have (by a like reaſoning as was uſed before, in determining the logarithm of the ratio of $b + x$ to b,) the fluxion of the aſymptotic area that meaſures the ratio of b to $b - x$ to the parallelogram of the hyperbola, or to aa, as $\dot{x}$ to $b - x$; whence it follows that that fluxion will be $= \frac{aa\dot{x}}{b-x} = \frac{aa\dot{x}}{b} + \frac{aax\dot{x}}{bb} + \frac{aax^2\dot{x}}{b^3} + \frac{aax^3\dot{x}}{b^4} + \frac{aax^4\dot{x}}{b^5} + \frac{aax^5\dot{x}}{b^5} + \frac{aax^6\dot{x}}{b^7} +$ &c. and conſequently, taking the fluents on both ſides, the aſymptotic area that is the logarithm of the ratio of b to $b - x$ will be equal to the ſeries $\frac{aax}{b} + \frac{aaxx}{2bb} + \frac{aax^3}{3b^3} + \frac{aax^4}{4b^4} + \frac{aax^5}{5b^5} + \frac{aax^6}{6b^6} + \frac{aax^7}{7b^7} +$ &c.

And conſequently, in the 6th place, if the hyperbola is equilateral, and b is taken $= a$, the logarithm of the ratio of (b to $b - x$, or) a to $a - x$ will be $= ax + \frac{xx}{2} + \frac{x^3}{3a} + \frac{x^4}{4a^2} + \frac{x^5}{5a^3} + \frac{x^6}{6a^4} + \frac{x^7}{7a^5} +$ &c.

And hence, in the 7th place, we may derive another ſeries for determining the logarithm of the ratio of 2 to 1. For if x is $= \frac{a}{2}$, the ratio of a to $a - x$ will be equal to the ratio of 2 to 1, and the foregoing ſeries will be $= \frac{aa}{2} + \frac{aa}{2.4} + \frac{aa}{3.8} + \frac{aa}{4.16} + \frac{aa}{5.32} + \frac{aa}{6.64} + \frac{aa}{7.128} +$ &c. which will therefore be the logarithm of the ratio of 2 to 1 in an hyperbola whoſe parallelogram is $= aa$.

And 8*thly*, 'Tis evident from hence that Napier's logarithm of 2, or of the ratio of 2 to 1, is $= \frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} + \frac{1}{6.64} + \frac{1}{7.128} +$ &c. for this is the value of the laſt-mentioned ſeries, when aa, or the parallelogram of the hyperbola, is call'd 1, which determines the logarithms to be of Napier's ſyſtem.

9*thly*, It may hence be obſerved, that the ſeries $\frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} + \frac{1}{6.64} + \frac{1}{7.128} +$ &c. is $=$ the ſeries $\frac{1}{2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} + \frac{1}{11.12} + \frac{1}{13.14} +$ &c. each of them being equal to Napier's logarithm of the ratio of 2 to 1.

10*thly*, It follows from theſe concluſions that the logarithm of the ratio of $a+x$ to $a-x$, (being equal to the ſum of the two logarithms of the ratios of $a+x$ to a, and of a to $a-x$) in an hyperbola whoſe parallelogram is $= aa$, is $= 2ax + \frac{2x^3}{3a} + \frac{2x^5}{5a^3} + \frac{2x^7}{7a^5} + \frac{2x^9}{9a^7} + \frac{2x^{11}}{11a^9} +$ &c.

by the help of which ſeries the logarithm of any propoſed ratio whatſoever may be computed; for the ſeries converges as long as x is not greater than a, and the quantities $a+x$ and $a-x$ may be taken in any proportion whatſoever to each other, without ſuppoſing x to be greater than a; and if x be but ſmall in compariſon of a, the ſeries will converge very ſwiftly,

Thus, if it be required to find the logarithm of the ratio of 2 to 1, we muſt put $a+x : a-x :: 2 : 1$, and we ſhall have $a+x=2a-2x$, whence $3x$ is $=a$, and x is $=\frac{a}{3}$. Therefore the logarithm of the ratio of 2 to 1, in an hyperbola whoſe parallelogram is aa, is $=\frac{2aa}{3}+\frac{2aa}{3\times27}+\frac{2aa}{5\times243}+\frac{2aa}{7\times2187}+\frac{2aa}{9\times19683}+\frac{2aa}{11\times717{,}147}+$ &c.

In like manner, to find the logarithm of the ratio of 3 to 1, we have $a+x : a-x :: 3 : 1$; whence $a+x=3a-3x$, and $4x=2a$, and conſequently $x=\frac{2a}{4}$ or $\frac{a}{2}$. Therefore the logarithm of the ratio of 3 to 1, in an hyperbola whoſe parallelogram is $=aa$, is $=\frac{2aa}{2}+\frac{2aa}{3\times8}+\frac{2aa}{5\times32}+\frac{2aa}{7\times128}+\frac{2aa}{9\times512}+\frac{2aa}{11\times2048}+$ &c.

Again, to find the logarithm of the ratio of 5 to 4, we have $a+x : a-x :: 5 : 4$; whence $4a$

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$+4 = 5a - 5x$, and $x = \frac{a}{9}$. Therefore the logarithm of this ratio is = to the ſeries $\frac{2aa}{9} + \frac{2aa}{3 \times 729} + \frac{2aa}{5 \times 59{,}049} + \frac{2aa}{7 \times 5{,}282{,}969} + \&c.$ which, 'tis evident, converges with great ſwiftneſs.

And in general, to find the logarithm of the ratio of any two given numbers m and n, we muſt ſay as $m : n :: a + x : a - x$; and we ſhall thence have $m \times \overline{a - x} = n \times \overline{a + x}$, or $ma - mx = na + nx$; whence $mx + nx = ma - na$, or $\overline{m + n} \times x = \overline{m - n} \times a$, and $x = \frac{m - n}{m + n} \times a$, which being ſubſtituted inſtead of x in the ſeries $2ax + \frac{2x^3}{3a} + \frac{2x^5}{5a^3} + \frac{2x^7}{7a^5} + \&c.$ will give the logarithm of the propoſed ratio in an hyperbola whoſe parallelogram is $= aa$.

If x is not ſufficiently ſmall in compariſon of a, to make the ſeries converge ſwiftly, it will be proper to uſe ſome artifice in obtaining the logarithm of the ratio propoſed, rather than to calculate it in a direct manner. Thus, if we would find the logarithm of the ratio of 10 to 1, we ſhall have $x = \frac{10 - 1}{10 + 1} \times a = \frac{9}{11} \times a$, the powers of which converge very ſlowly. We muſt therefore not attempt to find this logarithm directly by the ſeries, but muſt rather divide the propoſed ratio of 10 to 1, into two, or more, other ratios, whoſe logarithms may be more eaſily computed; and by computing, and then adding together, the logarithms of theſe component ratios, we ſhall obtain the logarithm required. Thus, if we divide the ratio of 10 to 1 into the two ratios of 10 to 8 and 8 to 1, the latter of which is triple of the ratio of 2 to 1, and therefore its logarithm is triple of the logarithm of the ratio of 2 to 1, and then compute the logarithms of theſe ratios, and to three times the logarithm of the ratio of 2 to 1 add the logarithm of the ratio of 10 to 8, or 5 to 4, the ſum will be equal to the logarithm of the propoſed ratio of 10 to 1. And the like muſt be done in other caſes.

11*thly*, If we ſubſtitute r and s inſtead of a and x reſpectively in the foregoing ſeries $2ax + \frac{2x^3}{3a} + \frac{2x^5}{5a^3} + \frac{2x^7}{7a^5} +$ &c. it will be converted into $2rs + \frac{2s^3}{3r} + \frac{2s^5}{5r^3} + \frac{2s^7}{7r^5} +$ &c. which will therefore be the logarithm of the ratio of $r+s$ to $r-s$ in an hyperbola whoſe parallelogram is rr. Therefore half this ſeries will be the logarithm of half this ratio, or of half the ratio of $\overline{r+s} \times \overline{r+s}$ to $\overline{r-s} \times \overline{r+s}$, or to $rr-ss$, and therefore of the ratio of $r+s$ to $\sqrt{rr-ss}$; that is, the ſeries $rs + \frac{s^3}{3r} + \frac{s^5}{5r^3} + \frac{s^7}{7r^5} + \frac{s^9}{9r^7} + \frac{s^{11}}{11r^9}$

+

$+ \frac{s^{13}}{13r^{11}} +$ &c. is the logarithm of the ratio of $r+s$ to $\sqrt{rr-ss}$ in an hyperbola whoſe parallelogram is rr, agreeably to what was ſhewn in art. 343.

A ſeries for expreſſing the external area of an equilateral hyperbole, included between the two axes and the curve, and a line drawn parallel to the tranſverſe axis.

346. Having thus gone thro' the principal ſpecies relating to the aſymptotic areas of an hyperbola, it will not be amiſs, before we quit this figure, to give the following ſeries for the determination of its external area, included between the curve and its two axes, and a line drawn parallel to the tranſverſe axis, as it is an extremely ſimple one in the law of the generation of its terms, and follows naturally from ſome of the foregoing articles.

It appears by art. 281, that if Dg (*Fig.* 82) be a rectangular hyperbola whoſe vertex is D, center a, and ſecond axis ax, and the ſemiaxis aD be equal to the radius CA of the circle CAB in *Fig.* 78, and og be drawn parallel to the tranſverſe axis aD, the fluxion, or infinitely ſmall increment, of the external area aDgo, included between the two axes aD, ao, the curve Dg, and the right line og, will be equal to the rectangle under the contemporary ſmall increment of the tangent AO and the ſecant CO; that is, if the radius CA be put $=r$, the ſine NS $=s$, and it's fluxion $=\dot{s}$, the fluxion of the area *aDgo* will be $\left(= \frac{\mathrm{MO} \times \mathrm{CM}}{r}\right.$, by *Coroll.* 3. $\left.\frac{r^3\dot{s}}{\overline{rr-ss}^{\frac{3}{2}}} \times \mathrm{CM}\right) = \frac{r^3\dot{s}}{\overline{rr-ss}^{\frac{3}{2}}} \times \frac{rr}{\sqrt{rr-ss}} = \frac{r^5\dot{s}}{\overline{rr-ss}^{2}} = r\dot{s} + \frac{2s^2\dot{s}}{r} + \frac{3s^4\dot{s}}{r^3} + \frac{4s^6\dot{s}}{r^5} + \frac{5s^3\dot{s}}{r^7} + \frac{6s^{10}\dot{s}}{r^9} + \frac{7s^{12}\dot{s}}{r^{11}} +$ &c. Therefore taking the fluents, the external area *aDgo*, whoſe baſe is equal to the tangent AO of the arc of which s is the ſine, in the circle whoſe diameter is equal to the axis of the hyperbola, is $= rs + \frac{2s^3}{3r} + \frac{3s^5}{5r^3} + \frac{4s^7}{7r^5} + \frac{5s^9}{9r^7} + \frac{6s^{11}}{11r^9} + \frac{7s^{13}}{13r^{11}} +$, &c. in which the law of continuation is very manifeſt.

A ſeries for determining the arch of a parabola.

347. By art. 285 it appears, that if the parameter of the parabola GVH, in *Fig.*83, be equal to the ſemiaxis *a*D, and the ordinate GH to the axis of that parabola be equal to the baſe *ao* of the external hyperbolic area *aDgo*, the rectangle under the parabolic arch GVH and the parameter, or ſemiaxis *a*D will be equal to the hyperbolic area *aDgo*. Conſequently the parabolic arch GVH is $= \frac{aDgo}{a\mathrm{D}} = \frac{aDgo}{\mathrm{CA}} = \frac{aDgo}{r} = s + \frac{2s^3}{3rr} + \frac{3s^5}{5r^4} + \frac{4s^7}{7r^6} + \frac{5s^9}{9r^8} + \frac{6s^{11}}{11r^{10}} + \frac{7s^{13}}{13r^{12}} +$ &c. that is, if the parameter of any parabola GV be call'd r, and the ſine of the circular arc of which the whole ordinate GH is the tangent,

gent, in a circle whereof r, or the parameter of the parabola, is the radius, be call'd s, the parabolic arch GVH will be equal to the foreſaid ſeries $s + \frac{2s^3}{3rr} + \frac{3s^5}{5r^4} + \frac{4s^7}{7r^6} + \frac{5s^9}{9r^8} + \frac{6s^{11}}{11r^{10}} + \frac{7s^{13}}{13r^{12}} +$ &c. which ſeries always converges, of whatever magnitude the parabolic arch be taken.

A ſeries expreſſing the relation of two ſemiordinates to the axis of an equilateral hyperbola, when the ſector corresponding to the greater ſemiordinate is any given multiple of the ſector correſponding to the leſſer ſemiordinate.

348. This ſeries bears ſo great a reſemblance to that which has been already given in art. 339. for expreſſing the relation between the ſines of two circular arcs whereof the greater arc is any given multiple of the leſſer, that it ought by no means to be omitted. Conceive therefore two ſemiordinates to the axis aD of the equilateral hyperbola Dy, in *Fig.* 82, to be drawn from the points b and g; and two other lines to be drawn from the ſame points to a the center of the hyperbola; and let the hyperbolic ſector aDg, correſponding to the ſemiordinate drawn from g, contain the hyperbolic ſector aDb, correſponding to the ſemiordinate drawn from b, n times; and let bf and go be parallel to aD. Then, 'tis evident, ao and af will be reſpectively equal to the ſemiordinates drawn from the points g and b.

Now let aD be put $= a$, $af = y$, and $ao = v$. And we ſhall have $bf = \sqrt{aa+yy}$ and $go = \sqrt{aa+vv}$. Therefore, if $\dot{v}$ be put for the fluxion of the line ao or v, the fluxion of the area aDgo will be $= \dot{v} \times \sqrt{aa+vv} = a\dot{v} + \frac{v^2\dot{v}}{2a} - \frac{v^4\dot{v}}{8a^3} - \frac{v^6\dot{v}}{16a^5} - \frac{5v^8\dot{v}}{128a^7} - \&c.$ and conſequently the area aDgo is $= av + \frac{v^3}{3.2a} - \frac{v^5}{5.8a^3} - \frac{v^7}{7.16a^5} - \frac{5v^9}{9.128a^7} - \&c.$ Alſo, if $g\alpha$ cuts the aſymptote aR in α, we ſhall have $o\alpha = ao$, and $g\alpha$ $(= og - o\alpha) = og - ao = \sqrt{aa+vv} - v$, and conſequently $g\alpha^2 = aa + 2vv - 2v\sqrt{aa+vv}$.

Draw DN and gR at right angles to the aſymptote aR, and cutting it in N and R. And by art. 190, the aſymptotic area DNRg will be equal to the ſector aDg. Therefore the ſector aDg is (= aſymptotic area DNRg) = area aDgo − triangle $a\alpha o$ − triangle aDN + triangle αgR $= \text{area } a\text{D}go - \frac{ao^2}{2} - \frac{a\text{D}^2}{4} + \frac{\alpha g^2}{4} = \text{area } a\text{D}go - \frac{vv}{2} - \frac{aa}{4} + \frac{aa}{4} + \frac{vv}{2} - \frac{v\sqrt{aa+vv}}{2} = \text{area } a\text{D}go - \frac{v\sqrt{aa+vv}}{2} = av + \frac{v^3}{3.2a} - \frac{v^5}{5.8a^3} - \frac{v^7}{7.16a^5} - \frac{5v^9}{9.128a^7} - \&c. - \frac{av}{2} - \frac{v^3}{4a} + \frac{v^5}{16a^3} + \frac{v^7}{32a^5} + \frac{5v^9}{256a^7} + \&c. = \frac{av}{2} - \frac{v^3}{12a} + \frac{3v^5}{80a^3} + \frac{5v^7}{224a^5} + \frac{35v^9}{2304a^7} + \&c.$

By the ſame kind of reaſoning we ſhall find the ſector xDb to be $= \frac{ay}{2} - \frac{y^3}{12a} + \frac{3y^5}{80a^3} + \frac{5y^7}{224a^5} + \frac{35y^9}{2304a^7} +$ &c.

But it was ſuppoſed that the former ſector was equal to n times the latter. Therefore $\frac{av}{2} - \frac{v^3}{12a} + \frac{3v^5}{80a^3} +$ &c. is $= \frac{nay}{2} - \frac{ny^3}{12a} + \frac{3ny^5}{80a^3} +$ &c. or $\frac{av}{2} - \frac{v^3}{12a} + \frac{3v^5}{80a^3} +$ &c. $- \frac{nay}{2} + \frac{ny^3}{12a} - \frac{3ny^5}{80a^3} -$ &c. is $= 0$; and, dividing all the terms by $\frac{a}{2}$, we ſhall have $v - \frac{v^3}{6aa} + \frac{3v^5}{40a^4} +$ &c. $- ny + \frac{ny^3}{6aa} - \frac{3ny^5}{40a^4} -$ &c. $= 0$.

Aſſume therefore $v = Ay + By^3 + Cy^5 +$ &c. and by ſubſtituting this ſeries inſtead of v in the foregoing equation, we ſhall have

$$\left.\begin{array}{l} Ay + By^3 + Cy^5 + \text{\&c.} \\ \quad - \frac{A^3y^3}{6aa} - \frac{3A^2By^5}{6aa} - \text{\&c.} \\ \qquad\qquad + \frac{3A^5y^5}{40a^4} + \text{\&c.} \\ - ny + \frac{ny^3}{6aa} - \frac{3ny^5}{40a^4} - \text{\&c.} \end{array}\right\} = 0.$$

Therefore $Ay - ny$ is $= 0$; whence A is $= n$. And $By^3 - \frac{A^3y^3}{6aa} + \frac{ny^3}{6aa}$ is $= 0$; whence B is $= \frac{A^3 - n}{6aa} = \frac{n^3 - n}{6aa} = n \times \frac{\overline{nn-1}}{2.3.aa}$

And 3dly, $Cy^5 - \frac{3A^5By^5}{6aa} + \frac{3A^5y^5}{40a^4} - \frac{3ny^5}{40a^4}$ is $= 0$; whence C is $= \frac{3n}{40a^4} + \frac{3A^2B}{6aa} - \frac{3A^5}{40a^4} = \frac{3n}{40a^4} + \frac{3n^2B}{6aa} - \frac{3n^5}{40a^4} = \frac{3n}{40a^4} + \frac{3n^2}{6aa} \times \frac{n^3-n}{6aa} - \frac{3n^5}{40a^4} = \frac{3n-3n^5}{40a^4} + \frac{n^5-n^3}{12a^4} = \frac{9n-9n^5}{120a^4} + \frac{10n^5-10n^3}{120a^4} = \frac{n^5-10n^3+9n}{120a^4} = n \times \frac{\overline{nn-1}}{2.3.a^2} \times \frac{\overline{nn-9}}{4.5.a^2}$. And in the ſame manner we may determine as many of the following coefficients D, E, F, &c. as we think fit.

Therefore v is $= ny + n \times \frac{\overline{nn-1}}{2.3.a^2} \times y^3 + n \times \frac{\overline{nn-1}}{2.3.a^2} \times \frac{\overline{nn-9}}{4.5.a^2} \times y^5 +$ &c. or $ny + \frac{\overline{nn-1}}{2.3.a^2} \times Ay^3 + \frac{\overline{nn-9}}{4.5.a^2} \times By^5 + \frac{\overline{nn-25}}{6.7.a^2} \times Cy^7 + \frac{\overline{nn-49}}{8.9.a^2} \times Dy^9 + \frac{\overline{nn-81}}{10.11.a^2} \times Ey^{11} +$ &c. in which the letters A, B, C, D, E, &c. denote, as they did in the courſe of the inveſtigation, the coefficients of the ſeveral terms; or, if theſe letters denote the whole terms, (A being $= ny$, and B to $n \times \frac{\overline{nn-1}}{2.3.a^2} \times y^3$, and ſo of the reſt) we ſhall have $v = ny + \frac{\overline{nn-1}}{2.3.a^2} \times Ay$

+

$+\frac{nn-9}{4.5.a^2}\times By^2+\frac{nn-25}{6.7.a^2}\times Cy^2+\frac{nn-49}{8.9.a^2}\times Dy^2+\frac{nn-81}{10.11.a^2}\times Ey^2+$ &c. in which the law of continuation is very manifeſt.

This ſeries was invented by Mr. De Moivre, and applied by him to the ſolution of certain algebraic equations of the fifth, ſeventh, and other odd powers. See the *Miſcellanea Curioſa*, vol. I. page 365, and De Moivre's *Miſcellanea Analytica*, page 14.

A remarkable analogy between circular ſectors and hyperbolic ſectors.

349. The ſeries $\frac{av}{2}-\frac{v^3}{12a}+\frac{3v^5}{40a^3}+\frac{5v^7}{224a^5}+\frac{35v^9}{2304a^7}+$ &c. found in the courſe of the foregoing article for determining the hyperbolic ſector *a*D*g* from its correſponding ſemiordinate *ao*, or *v*, is exactly ſimilar to, or has the ſame coefficients with, the ſeries by which a ſector of a circle may be determined from its ſine, that is, from the ſine of the arc that is its baſe. For ſuch a circular ſector, being equal to a right-angled triangle whoſe baſe is the arc of the ſector and whoſe height is the radius, or to a rectangular parallelogram whoſe baſe is the arc of the ſector, and whoſe height is half

the radius, will be (by art. 329.)

$$= \frac{r}{2} \times \overline{s + \frac{s^3}{2.3.r^2} + \frac{3s^5}{2.4.5.r^4} + \frac{3.5.s^7}{2.4.6.7.r^6} + \frac{3.5.7.s^9}{2.4.6.8.9r^8} + \&c.} = \frac{rs}{2} + \frac{s^3}{12r} + \frac{3s^5}{80r^3} + \frac{5s^7}{224r^5} + \frac{35s^9}{2304a^7} + \&c.$$

which has the ſame coefficients with the ſeries for the hyperbolic ſector, and differs from it only in that it has all its terms affirmative, whereas in the hyperbolic ſeries the ſecond term is negative. And this is another inſtance of the great analogy that is found to be between the circle and the rectangular hyperbola, according to what has been already hinted in art. 193 and 230.

Of the ſerieſes relating to the Logarithmic curve.

Of the ſerieſes for expreſſing an abſciſs of the axis of a logarithmic curve, in powers of the difference of the two ordinates that bound it, one of thoſe ordinates themſelves, and the ſubtangent; or according to the phraſes commonly uſed, for finding the logarithm of a given number.

350. If in the logarithmic curve fAC (*Fig.* 73) we take any two ordinates $\mu\pi$ and AO, whereof the greater AO is not more than double of the leſſer $\mu\pi$, and call the ſubtangent of this curve a, the leſſer of theſe ordinates b, and the

greater $b+x$, the abſciſs πO of the axis of this curve intercepted between theſe two ordinates, or the logarithm of their ratio to each other, will be $= \frac{ax}{b} - \frac{ax^2}{2bb} + \frac{ax^3}{3b^3} - \frac{ax^4}{4b^4} + \frac{ax^5}{5b^5} - \frac{ax^6}{6b^6} +$ &c. &c.

For if this abſciſs, or logarithm, be called z, and its fluxion, or infinitely ſmall increment, $\dot z$, and the fluxion, or contemporary increment, of the ordinate $b+x$ be call'd $\dot x$, we ſhall have, by art. 200, $\dot x : \dot z :: b+x : a$; whence $\dot z$ is $= \frac{a\dot x}{b+x} = \frac{a\dot x}{b} - \frac{ax\dot x}{bb} + \frac{ax^2\dot x}{b^3} - \frac{ax^3\dot x}{b^4} + \frac{ax^4\dot x}{b^5} - \frac{ax^5\dot x}{b^6} +$ &c. and conſequently, taking the fluents, z will be $= \frac{ax}{b} - \frac{ax^2}{2bb} + \frac{ax^3}{3b^3} - \frac{ax^4}{4b^4} + \frac{ax^5}{5b^5} - \frac{ax^5}{6b^6} +$ &c. which converges as long as x is not greater than b, or as long as $b+x$ is not more than double of b.

If b, or $\mu\pi$, the leſſer ordinate, be taken equal to a, or the ſubtangent, the ſeries will be $x - \frac{x^2}{2a} + \frac{x^3}{3aa} - \frac{x^4}{4a^3} + \frac{x^5}{5a^4} - \frac{x^6}{6a^5} +$ &c. which is therefore the logarithm of the ratio of $a+x$ to a in a logarithmic curve whereof a is the ſubtangent.

351. If it be required to find the logarithm of the ratio of b to $b-x$ in a logarithmic curve

whoſe ſubtangent is a, we muſt make uſe of the following proportion, to wit, $\dot{x} : \dot{z} :: b-x : a$; whence $\dot{z}$ is $= \frac{a\dot{x}}{b-x} = \frac{a\dot{x}}{b} + \frac{ax\dot{x}}{bb} + \frac{ax^2\dot{x}}{b^3} + \frac{ax^3\dot{x}}{b^4} + \frac{ax^4\dot{x}}{b^5} + \frac{ax^5\dot{x}}{b^6} +$ &c. and conſequently z is $= \frac{ax}{b} + \frac{ax^2}{2bb} + \frac{ax^3}{3b^3} + \frac{ax^4}{4b^4} + \frac{ax^5}{5b^5} + \frac{ax^6}{6b^5} +$ &c.

If $b = a$, or the greater of the two ordinates is equal to the ſubtangent, we ſhall have z, or the logarithm of the ratio of a to $a-x$ in the logarithmic curve whoſe ſubtangent is b, $= x + \frac{x^2}{2a} + \frac{x^3}{3aa} + \frac{x^4}{4a^3} + \frac{x^5}{5a^4} + \frac{x^6}{6a^5} \pm$ &c.

352. It follows from theſe determinations that the logarithm of the ratio of $a+x$ to $a-x$ (being equal to the ſum of the logarithms of the ratios of $a+x$ to a, and of a to $a-x$) is $= 2x + \frac{2x^3}{3aa} + \frac{2x}{5a^4} + \frac{2x^7}{7a^6} + \frac{2x^9}{9a^8} + \frac{2x^{11}}{11a^{10}} +$ &c.

Of the ſerieſes for expreſſing the difference of two ordinates in a logarithmic curve in powers of the abſciſs of the axis intercepted between them, and of the ſubtangent; or, according to the phraſes commonly uſed, for finding the number from the logarithm.

353. Having ſhewn how the logarithm of a given ratio, or the abſciſs of the axis of a loga-

rithmic curve, intercepted between two given ordinates, may be found from thoſe ordinates, and expreſſed in powers of their difference and of the ſubtangent, it remains that we mention the ſerieſes by which the ratio of the ordinates may be diſcovered from the abſciſs, or logarithm, given, or their difference (one of them being ſuppoſed to be equal to the ſubtangent) expreſſed in powers of the abſciſs and ſubtangent. Now this may be done by reverting the foregoing ſerieſes in the ſame manner as was before done to the ſerieſes derived from the circle and hyperbola.

354. Since therefore the ſeries $x - \frac{x^2}{2a} + \frac{x^3}{3aa} +$ &c. is $=$ log. $\frac{a+x}{a} = z$, aſſume $x = Az + Bz^2 + Cz^3 +$ &c. and we ſhall have

$$\left.\begin{array}{llll} Az & + Bz^2 & + Cz^3 & + \&c. \\ & -\frac{A^2z^2}{2a} & -\frac{2ABz^3}{2a} & - \&c. \\ -z & & +\frac{A^3z^3}{3aa} & \end{array}\right\} = 0.$$

Therefore $Az - z$ is $= 0$; and $A = 1$. 2dly, $Bz^2 - \frac{A^2z^2}{2a}$ is $= 0$; whence $B = \frac{A^2}{2a} = \frac{1}{2a}$. 3dly, $Cz^3 - \frac{2ABz^3}{2a} + \frac{Az^{3}{}^{3}}{3aa}$ is $= 0$; whence C is $= \frac{2AB}{2a} - \frac{A^3}{3aa} \left(= \frac{B}{a} - \frac{1}{3aa} = \frac{1}{2aa} - \frac{1}{3aa} \right) = \frac{1}{2.3.aa}$. And in the ſame manner we ſhall find D to be $= \frac{1}{2.3.4.a^3}$, E to be $= \frac{1}{2.3.4.5.a^4}$, and ſo of the reſt.

Therefore x, or the difference of the two ordinates a and $a + x$ whoſe abſciſs is z, in a logarithmic curve, whoſe ſubtangent is a, is $= z + \frac{z^2}{2a} + \frac{z^3}{2.3.aa} + \frac{z^4}{2.3.4.a^3} + \frac{z^5}{2.3.4.5.a^4} +$ *&c.*

355. And by a like proceſs it will appear that if z be put for the logarithm of the ratio of a to $a - x$, in a logarithmic curve whoſe ſubtangent is a, (in which caſe it has been ſhewn that z is $= x + \frac{x^2}{2a} + \frac{x^3}{3aa} +$ *&c.* without any negative terms) the difference x of theſe ordinates will be $= z - \frac{z^2}{2a} + \frac{z^3}{2.3\,aa} - \frac{z^4}{2.3.4.a^3} + \frac{z^5}{2.3.4.5.a^4}$ *&c.* in which ſeries the terms are alternately affirmative and negative.

356. Hence 'tis evident that, if z is $= a$, or the given logarithm, or abſciſs of the axis, is equal to the ſubtangent, the firſt of theſe ſerieſes will be in this caſe $= a + \frac{a}{2} + \frac{a}{2.3} + \frac{a}{2.3.4} + \frac{a}{2.3.4.5} +$ *&c.* and the ſecond ſeries will be $a - \frac{a}{2} + \frac{a}{2.3} - \frac{a}{2.3.4} + \frac{a}{2.3.4.5} +$ *&c.* Therefore

the proportion of $a+x$ to a in this caſe will be equal to that of $a+a+\frac{a}{2}+\frac{a}{2.3}+\frac{a}{2.3.4}+\frac{a}{2.3.4.5}+$ &c. to a, or of $1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\frac{1}{2.3.4.5}+$ &c. to 1, or of 2.718,281,828,459, &c. to 1; and the proportion of a to $a-x$ will be equal to that of a to $a-a+\frac{a}{2}-\frac{a}{2.3}+\frac{a}{2.3.4}-\frac{a}{2.3.4.5}+$ &c. or of 1 to $1-1+\frac{1}{2}-\frac{1}{2.3}+\frac{1}{2.3.4}-\frac{1}{2.3.4.5}+$ &c. or to $\frac{1}{2}-\frac{1}{2.3}+\frac{1}{2.3.4}-\frac{1}{2.3.4.5}+$ &c. or of 1 to .367,879,441,171, &c.

Therefore, according to Mr. Cotes's manner of expreſſion, the *ratio modularis* is that of 2.718,281,828,459, &c. to 1, or that of 1 to .367,879,441,171, &c. agreeably to art. 203 and 221.

Theſe are the principal ſerieſes relating to the ſubject of the foregoing pages; and with theſe I therefore conclude the book, and take my final leave of the reader.

FINIS.

ERRATA.

IN Page 7, inſtead of $\frac{17}{60}\times\frac{40}{60\times60}$ read $\frac{17}{60}+\frac{40}{60\times50}$. In page 9, in the laſt operation of the extraction of the ſquare-root there ſet down, inſtead of 69282016, read 69294016; and inſtead of 655484, read 633484. In page 26, prop. 7, make a reference to fig. 7. In page 27, line 6 from the bottom, inſtead of *leſs*, read *leſs and leſs*. In page 63, line 3, inſtead of $\times BE^q$, read $+ BE^q$; and the ſame in line 5. And, in the 5th line of prop. 22, put a ſemi-colon after the words *Take* AD = AB; and after the ſemi-colon write the word *and*. In page 64. line 18, inſtead of CP × RP, write CP + RP; and the ſame in line 19. And in line 20, in two places, inſtead of CA×AD, write CA+AD. In page 66, line 6, inſtead of S, read SO. In page 71, there ought to be no break in the paragraph after the words, *contrary way*. In page 74, line 1, inſtead of *tangets*, read *tangents*. In page 79, at the end of the ſecond paragraph, inſtead of fig. 33, read fig. 38; and in line 6, from the bottom, inſtead of fig. 32, read fig. 37. In page 123. line 7, from the bottom, inſtead of *Iſaac Newton*, read *Sir Iſaac Newton*. In page 143, at the end of the 1ſt line of art. 132, dele the word *of*. In page 163, line 3, of Corol. 2. after the word *center*, inſtead of a ſmall c write a capital C. In page 167, line 11, before the word *previouſly*, prefix the figures 147, for the number of the article. In page 176, line 8, inſtead of; *it follows*, read *it follows that*. In page 199, line 15 from the bottom, after the word *equal* dele the mark :: In page 245, line 3. inſtead of *this corollary*, read *this diſſertation*. In page 266, line 11, inſtead of 10,000,100th, read 10,000,000th. In page 268, line 8, inſtead of *continued*, read *continued quantity*. In page 286, line 5 from the bottom, inſtead of *logarithms*, read *logarithm*; and line 4 from the bottom, inſtead of *logratbim*, read *logarithm*. In page 305, line 13, inſtead of *colums*, read *columns*. In page 312, line 6, inſtead of *garithms*, read *logarithms*. In page 313, line 10 from the bottom, inſtead of *diving*, read *dividing*. In page 372, line 4 from the bottom, dele the ſemicolon after the word *tangents*. In page 364, line 3 from the bottom, inſtead of “ *of an hyper-*, read “ *de-*. In page 425, line 15 from the bottom, at the end of the line after =, read *r*. In page 459, line 6, inſtead of *hyperbole*, read *hyperbola*.